4-vector, Lorentz transformation and de Broglies derivation of matter waves

1 Lorentz transformation for 4-momentum

In earlier articles, we saw that the energy and the momentum of a particle with rest mass m_0 and velocity $\vec{v}$ is given by

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \tag{1}$$

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \tag{2}$$

And, they satisfy the relation:

$$(m_0 c)^2 = \left(\frac{E}{c}\right)^2 - (\vec{p})^2 \tag{3}$$

Let's say that these values are the ones measured by an observer in a reference frame S. Now, let's ask the following question: With what value would the energy and momentum be measured by another observer moving with a constant velocity u along positive x-direction relative to the observer in S? In other words, what would be the Lorentz transformation for energy and momentum?

We can calculate this using Lorentz transformation of velocity explained in our earlier article. From Lorentz transformation, we have:

$$v'_x = \frac{v_x - u}{1 - \frac{v_x u}{c^2}} \tag{4}$$

$$v'_y = \frac{v_y}{\gamma(1 - \frac{v_x u}{c^2})} \tag{5}$$

$$v'_z = \frac{v_z}{\gamma(1 - \frac{v_x u}{c^2})} \tag{6}$$

Then, we have:

$$E' = \frac{m_0 c^2}{\sqrt{1 - (v'^2_x + v'^2_y + v'^2_z)/c^2}} \tag{7}$$

$$\vec{p'} = \frac{m_0 \vec{v'}}{\sqrt{1 - (v_x'^2 + v_y'^2 + v_z'^2)/c^2}} \tag{8}$$

This is a very complicated calculation. If one successfully does it, one gets (You don't need to do it):

$$\begin{aligned} p_x' &= \gamma \left(p_x - \frac{u}{c} \frac{E}{c} \right) && (9) \\ p_y' &= p_y && (10) \\ p_z' &= p_z && (11) \\ \frac{E'}{c} &= \gamma \left(\frac{E}{c} - \frac{u}{c} p_x \right) && (12) \\ \gamma &= \frac{1}{\sqrt{1 - u^2/c^2}} && (13) \end{aligned}$$

Remarkably, we had the *same* Lorentz transformation formula for position and time (a.k.a spacetime) as follows:

$$\begin{aligned} x' &= \gamma \left(x - \frac{u}{c} ct \right) && (14) \\ y' &= y && (15) \\ z' &= z && (16) \\ ct' &= \gamma \left(ct - \frac{u}{c} x \right) && (17) \\ \gamma &= \frac{1}{\sqrt{1 - u^2/c^2}} && (18) \end{aligned}$$

So, all the complicated calculations actually pay off! It's God's miracle. Is this agreement coincidence? One should suspect that there is a deep mechanism behind it that made all these possible. Let's look into this. In case of spacetime, Lorentz transformation preserves proper time as follows:

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y^2) - (\Delta z)^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y'^2) - (\Delta z')^2 \tag{19}$$

In case of energy and momentum (a.k.a 4-momentum), Lorentz transformation preserves the rest mass as follows

$$(m_0 c)^2 = \left(\frac{E}{c} \right)^2 - (\vec{p})^2 = \left(\frac{E'}{c} \right)^2 - (\vec{p'})^2 \tag{20}$$

To satisfy this condition, the Lorentz transformation for 4-momentum had to take the same form as the one for spacetime.

2 Wave number and angular frequency

If you learn some physics or math, you may know that a wave can be expressed as follows:

$$A \sin(k_x x + k_y y + k_z z - \omega t + \phi) \tag{21}$$

where the expression inside the parenthesis is called "phase," and k's are called wave numbers, and ω is called angular frequency. If we express the wave numbers as a vector as $\vec{k} = (k_x, k_y, k_z)$, it gives the direction to which the wave is propagating, and the wavelength is given by $\lambda = \frac{2\pi}{|\vec{k}|}$. Similarly, the period is given by $T = \frac{2\pi}{\omega}$.

Now, we want to obtain the Lorentz transformation for the wave number and angular frequency. To this end, notice that different observers must agree on the phase. If an event took place at a peak, different observer must agree that it took at the peak. Similarly, for a valley as well. This gives you the following condition:

$$k'_x x' + k'_y y' + k'_z z' - \frac{\omega'}{c}(ct') = k_x x + k_y y + k_z z - \frac{\omega}{c}(ct) \tag{22}$$

If we denote

$$\vec{K} = \begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega/c \end{pmatrix}, \qquad \vec{X} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \tag{23}$$

then, we have:

$$\begin{aligned}
(\vec{K}')^T \eta \vec{X}' &= (\vec{K})^T \eta \vec{X} && (24) \\
(\vec{K}')^T \eta \Lambda \vec{X} &= (\vec{K})^T \Lambda^T \eta \Lambda \vec{X} && (25) \\
(\vec{K}')^T &= (\vec{K})^T \Lambda^T && (26) \\
\vec{K}' &= \Lambda \vec{K} && (27)
\end{aligned}$$

So, we conclude that wave numbers and angular frequency transform as 4-vector! Of course, this was actually expected since $(k_x x + k_y y + k_z z - \frac{\omega}{c}(ct))$ can be understood as the dot-product of two 4-vectors. Summarizing we obtained:

$$\begin{aligned}
k'_x &= \gamma \left(k_x - \frac{u}{c}\frac{\omega}{c} \right) && (28) \\
k'_y &= k_y && (29) \\
k'_z &= k_z && (30) \\
\frac{\omega'}{c} &= \gamma \left(\frac{\omega}{c} - \frac{u}{c} k_x \right) && (31) \\
\gamma &= \frac{1}{\sqrt{1 - u^2/c^2}} && (32)
\end{aligned}$$

Problem 1. Obtain the relativistic Doppler formula $f = f_0\sqrt{\frac{1-u/c}{1+u/c}}$ using the above formulas.

3 De Broglie's derivation of matter waves

Long before de Broglie came up with his idea of matter waves, Planck had come up with the following relation.

$$E = hf \tag{33}$$

where E is the energy of photon, h is Planck's constant, and f is frequency of photon. Considering that the relation between frequency and angular frequency is given by $\omega = 2\pi f$, and $\hbar$(pronounced "h-bar") is defined by $h/(2\pi)$, we can express the energy of photon as follows:

$$E = \hbar\omega \tag{34}$$

Given this, de Broglie compared the formulas (9-13) and the formulas (28-32), and asserted:

$$\vec{p} = \hbar\vec{k} \tag{35}$$

and claimed that (34) and (35) are valid not only for photons, but also for *any* particles. Let's see what this implies. Using $\lambda = \dfrac{2\pi}{|\vec{k}|}$, (35) can be re-expressed as:

$$\lambda = \frac{h}{p} \tag{36}$$

In other words, a particle with momentum p has a wavelength given by $\dfrac{h}{p}$. Actually, he showed that one can derive this relation from Bohr's quantization principle as well. This should be all familiar if you read our earlier article "De Broglie's matter waves."