Ampere's law

In an earlier article, I have mentioned that an electric current induces magnetic field. In this article, we will see how much. This is dictated by Ampere's law as follows:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i \tag{1}$$

where the line integral is along a closed loop called "Amperian loop," and i is the net current enclosed by the loop. For an example, see Fig. 1.

The arrow indicates the direction of line integral. In this case, the net current is $i_1 - i_2$, since i_1 is going up and i_2 is going down. Now, let's calculate the magnetic field due to the current in a long straight wire. See Fig. 2. This is given by

$$\oint \vec{B} \cdot d\vec{s} = B \cdot 2\pi r = \mu_0 I \tag{2}$$

$$B = \frac{\mu_0 I}{2\pi r} \tag{3}$$

Now, let's express Ampere's law using curl. To this end, we need to express the electric current in terms of current density. If we denote the charge density as ρ, and the current density as $\vec{J}$, and use the results in "Continuity equation" and "Divergence and Stoke's theorem," we have:

$$\int \frac{\partial \rho}{\partial t} dV = -\oint \vec{J} \cdot d\vec{A} \tag{4}$$

If we consider a fixed region (i.e. the one that doesn't change over time) then, the above integral becomes:

$$-\frac{d \int \rho dV}{dt} = \oint \vec{J} \cdot d\vec{A} \tag{5}$$

As electric current "i" is defined by electric charge passing through a certain cross section per unit time, the above equation is $+i$. (We have the positive sign, since i must have the same sign as the right-hand side of the above equation.)

Now, let's apply Green's theorem to (1). We have:

$$\int (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \int \vec{J} \cdot d\vec{A} \tag{6}$$

Thus we conclude:

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{7}$$

Problem 1. See Fig. 3. We have three currents. i_1 and i_3 are directed out of the page and i_2 is directed into the page. Then, what would be $\int B \cdot d\vec{s}$ for the path denoted in the figure.

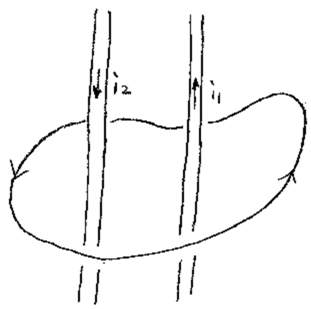

Figure 1: First case

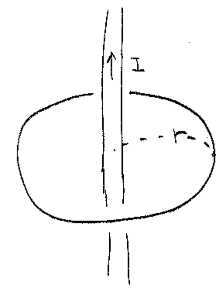

Figure 2: Long straight wire

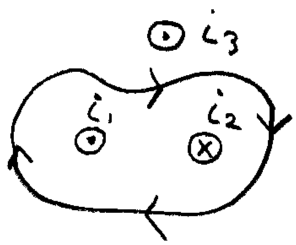

Figure 3: Problem 1

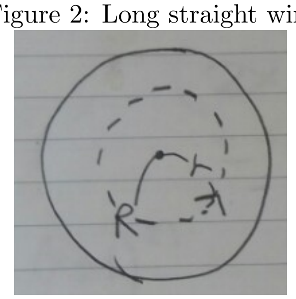

Figure 4: Problem 4

(Hint: the path is clockwise. Therefore, the answer would be negative one times what the answer would be if the path were anti-clockwise.)

Problem 2. See Fig. 1. in our earlier article "The force between two parallel wires through each of which electric current passes." If the distance between the two parallel wires are halved, how many times stronger would the force between them become?

Problem 3. See Fig. 4. for the cross section of a uniformly distributed current I that is directed out of the page inside an infinitely long straight wire of radius R. Notice that it has a cylindrical symmetry with respect to the center of the wire. If we take an Amperian loop with a circle with radius $r < R$ centered around the center of the wire, we can calculate the magnetic field there by considering that i, the current inside the Amperian loop is given by

$$i = \frac{\pi r^2}{\pi R^2} I \tag{8}$$

from the fact that the current is uniform. Then, from

$$\oint \vec{B} \cdot d\vec{s} = B \cdot 2\pi r = \mu_o i \tag{9}$$

we get

$$B = \frac{\mu_o I r}{2\pi R^3} \tag{10}$$

Similarly, find the magnetic field outside the wire (i.e. for $r > R$).

Summary

- $\nabla \times \vec{B} = \mu_0 \vec{J}$ i.e.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i$$