The Cartesian coordinate system and graph

To locate a point in a plane, you need two numbers: where the point is located horizontally, and where the point is located vertically. See Fig.1. We have two axes: the x-axis, which is horizontal, and the y-axis, which is vertical. For each axis, the numbers are labeled. The numbers labeled in x-axis increase as you go rightward and decrease as you go leftward. The numbers labeled in y-axis increase as you go upward and decrease as you go downward. The x-axis meets the y-axis at O, which is called the "origin," and is where both of the labeled numbers on two axes are zero. You can locate a point by the two numbers on the axes. For example, see P_1. Its x-coordinate is 2, since you meet at 2 on the x-axis if you vertically travel from P_1 to the x-axis. Also, its y-coordinate is 3, since you meet at 3 on the y-axis if you horizontally travel from P_1 to the y-axis. Altogether, we say the coordinates of P_1 is $(2, 3)$. Also, there is no reason why the coordinates have to be positive and integers. For example, P_2's coordinate is $(-3, 1.5)$. There are other examples: P_3's coordinate is $(-2.4, 2)$ and P_4's coordinate is $(1.7, -2.6)$. Given that we now know what the Cartesian coordinate system is, let's use it to draw graphs. For example, let's draw $y = x$. This means that the y coordinate is equal to the x coordinate. See Fig.2. The graph is a collection of such points. P_5 whose coordinates are $(1, 1)$ and P_6 whose coordinates are $(-3, -3)$ are two examples of such points. Notice also that the line passes through the origin as $y = 0$ when $x = 0$.

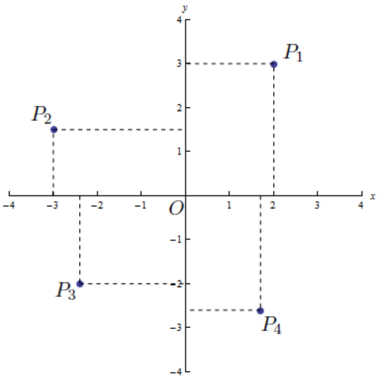

Figure 1: Cartesian coordinate system

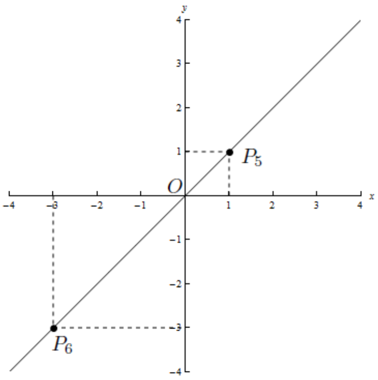

Figure 2: $y = x$

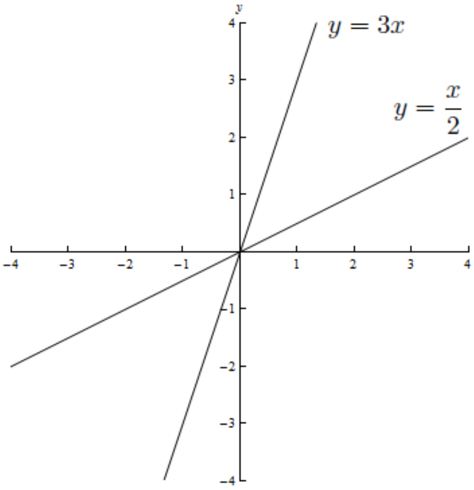

Figure 3: $y = 3x,\ \frac{x}{2}$

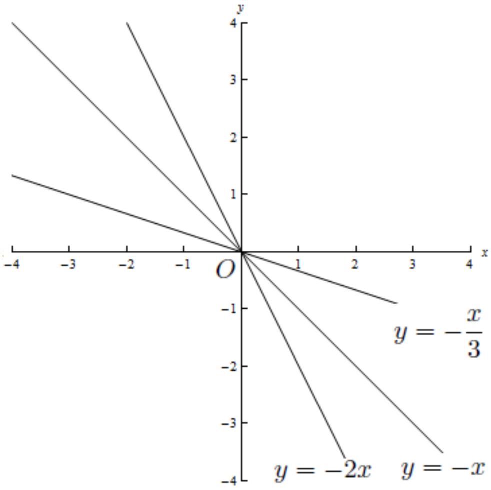

Figure 4: $y = -2x,\ -x,\ -\frac{x}{3}$

In Fig.3. you see other graphs. We have $y = 3x$, $y = \frac{x}{2}$. One can easily see that the graph $y = 3x$ is very steep while the graph $y = \frac{x}{2}$ is not steep. Mathematically, the slope is defined by the ratio of the y increase to the x increase. For example, let's calculate the slope of $y = 3x$. When x is equal to 0, y is also equal to 0. When x is equal to 2, y is equal to 6. While x increases by 2, y increases by 6. So 6 divided by 2 is 3. Therefore, the slope is 3. Similarly, we can see that the slope of the graph $y = \frac{x}{2}$ is $\frac{1}{2}$. More generally, the slope of the graph for $y = mx$ is given by m. Also, there is no reason why the slope has to be always positive. For example, see Fig.4. We have $y = -2x$, $y = -x$ and $y = -\frac{x}{3}$. The slopes are, respectively, -2, -1 and $-\frac{1}{3}$.

Notice that so far we have considered the graph of type $y = mx$. In other words, y divided by x (i.e. the ratio of y to x) is m. When y divided by x is constant, we say y is "proportional" to x. In such a case, if you double the x coordinate, the y coordinate is doubled. The ratio of y to x remains the same. Similarly, if you triple x, y is tripled. The ratio remains the same. If you halve x, y is halved. The ratio remains the same. Also, if x is 0, y is also 0 as m times 0 is always 0. Given this, let's consider a slightly different type of graph. See Fig.5. We have a graph $y = 2x$ the type we have already considered, and another type of graph: $y = 2x + 1$ and $y = 2x - 3$. We see that $y = 2x + 1$ is obtained by moving $y = 2x$ by 1 upward. Similarly, we see that $y = 2x - 3$ is obtained by moving $y = 2x$ by 3 downward.

Now, is y proportional to x in this other type of graph? The answer is no. Let's check $y = 2x + 1$ first. When $x = 1$ we have $y = 3$. When $x = 2$

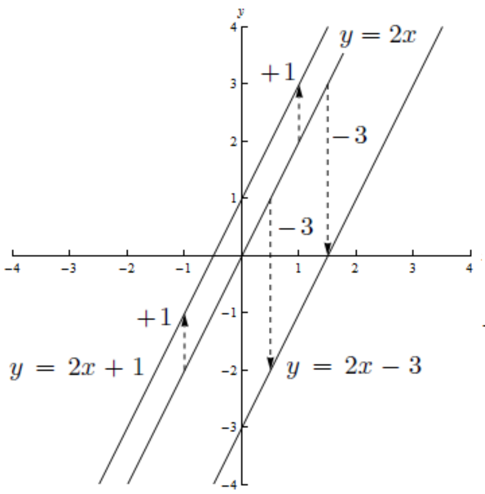

Figure 5: $y = 2x + 1,\ 2x,\ 2x - 3$

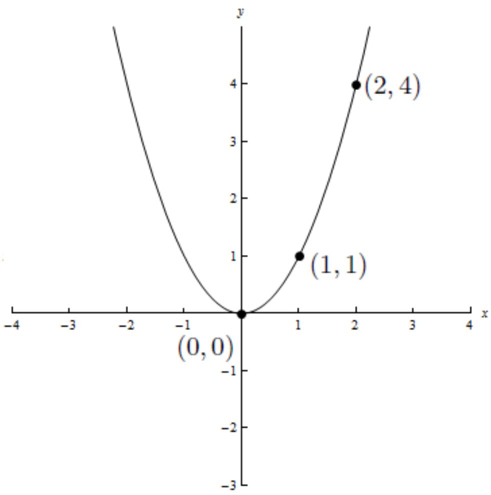

Figure 6: $y = x^2$

we have $y = 5$. While x is doubled, y is not doubled. Therefore, y is not proportional to x. Similarly, you can also check that y is not proportional to x if $y = 2x - 3$. The y coordinate is proportional to the x coordinate only in the case in which $y = mx$ is satisfied.

How about the slope of these graphs? Let's check $y = 2x + 1$ first. When x increased by 1 from 1 to 2, y increase by 2 from 3 to 5. Therefore, the slope is 2 as 2 divided by 1 is 2. Similarly, one can check that the slope of $y = 2x - 3$ is also 2. More generally, the slope of the graph of the form $y = mx + b$ is m. Let's check it. When $x = x_0$ we have $y_0 = mx_0 + b$. When x increases by Δx (pronounced "delta x") x becomes $x' = x_0 + \Delta x$ and y becomes

$$y' = m(x_0 + \Delta x) + b = mx_0 + b + m\Delta x = y_0 + m\Delta x \tag{1}$$

In other words, y increases by $m\Delta x$. So, the ratio of y increase to x increase is given by $\dfrac{m\Delta x}{\Delta x} = m$. This completes the proof.

Let me add a comment, at this point. We see that the slope, the ratio of y increase to x increase is constant (i.e. independent of Δx) for the graph of the form $y = mx + b$. However, for all the other types of graph, such as $y = x^2$, the slope is not constant and depends on where the points lie in the graph. See Fig.6. In this case, we have $y = 0$ for $x = 0$, $y = 1$ for $x = 1$. Therefore y increases by 1 while x increases by 1. So, the slope seems to be 1. On the other hand, we have $y = 4$ when $x = 2$. Therefore when y increases by 3 from $y = 1$ to $y = 4$ while x increases by 1 from $x = 1$ to $x = 2$. So, the slope seems to be 3. In other words, we see that the slope is not constant. Indeed, if you look at the graph carefully, you see that the line gets steeper and steeper for bigger x. How the slope can be calculated

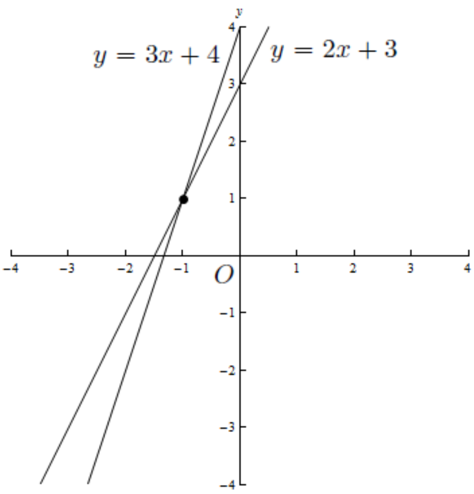

Figure 7: $y = 2x + 3,\ y = 3x + 4$

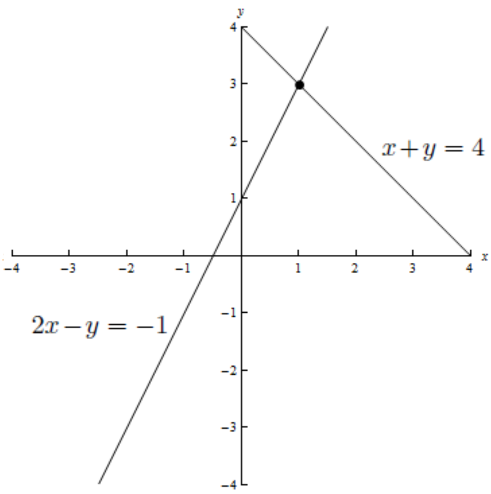

Figure 8: $2x - y = -1,\ x + y = 4$

in cases such as this is what you learn in the first day of class in "calculus," a very important subject in mathematics, which finds wide applications in science and engineering. We promise to talk about the slope in such cases when we teach you calculus in later articles.

We can also obtain a solution to an equation using the graph. For example, let's solve $2x + 3 = 3x + 4$. See Fig.7. We can solve this equation by drawing two graphs: $y = 2x + 3$ and $y = 3x + 4$, and then seeing where they meet. When they meet the equation is satisfied as we have $y = 2x + 3 = 3x + 4$. From the graph we see that they meet at $(-1, 1)$ (i.e. when $x = -1$ and $y = 1$.) So, the answer is $x = -1$.

Also, there are other ways to express the same graph slightly differently. For example, $y = 2x + 1$ is the same graph as $2x - y = -1$. We can see this as:

$$y = 2x + 1 \tag{2}$$

$$y - 1 - y = 2x + 1 - 1 - y \tag{3}$$

$$-1 = 2x - y \tag{4}$$

This suggests that we can solve equations with two unknowns using graph. For example, we can solve $2x - y = -1$, and $x + y = 4$ by drawing the graphs for both equations and finding the intersection. See Fig. 8. They intersect at (1,3). Therefore, the answer is $x = 1$ and $y = 3$.

So far, we had two axes: x-axis and y-axis. However, it is possible to add one more axis, the z-axis. Then we can actually describe a three dimensional space, instead of a plane as we did before. For example, we can locate a point in space by three numbers. See Fig.9. We see a point Q which has (1,2,4) as the coordinate.

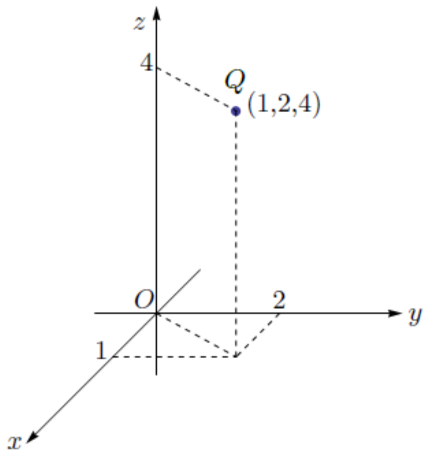

Figure 9: 3-dimensional Cartesian coordinate system

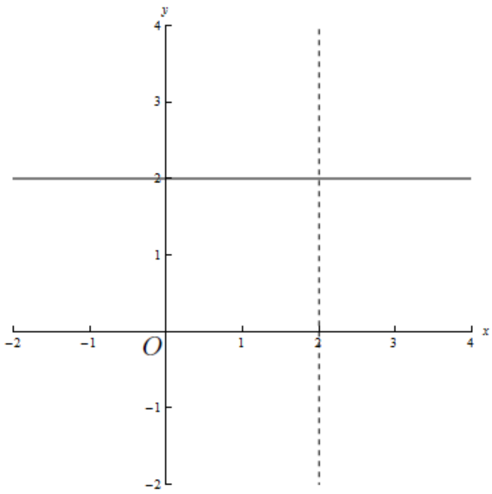

Figure 10: $x = 2,\ y = 2$

Final comment. The 17th century French philosopher René Descartes was known for often staying in his bed until noon. One day, while staying in his bed, he found a fly crawling on the ceiling. Then, he suddenly figured out how to tell someone else where the fly was. He found out that he could tell them how far the fly was from each wall. That's how he came up with the Cartesian coordinate, named after him.

Problem 1. Let's say that a graph of the form $y = mx + b$ passes these points:

$$(1,2),(4,4),(7,6),(10,8) \tag{5}$$

What are m and b? (Hint[1])

Problem 2. In Fig. 10 we have two graphs: one with a dotted line, and one with a solid line. One of them is $x = 2$ and the other $y = 2$. Which one is which?

Problem 3. Let's say that a graph of the form $y = mx + b$ passes (x_0, y_0). Express b in terms of x_0, y_0, and m. (Hint[2])

Problem 4. In Fig. 11 we have three graphs: A, B, C. They are $y = x^2, y = 2x^2, y = \frac{x^2}{2}$. Which one is which? Notice that y can be never negative for all three graphs as something squared multiplied by a positive number is always non-negative, and can be 0, which is the minimum only when x is 0.

[1] Find the slope m first by calculating how much y increases when x increases certain amount.

[2] Solve $y_0 = mx_0 + b$ for b.

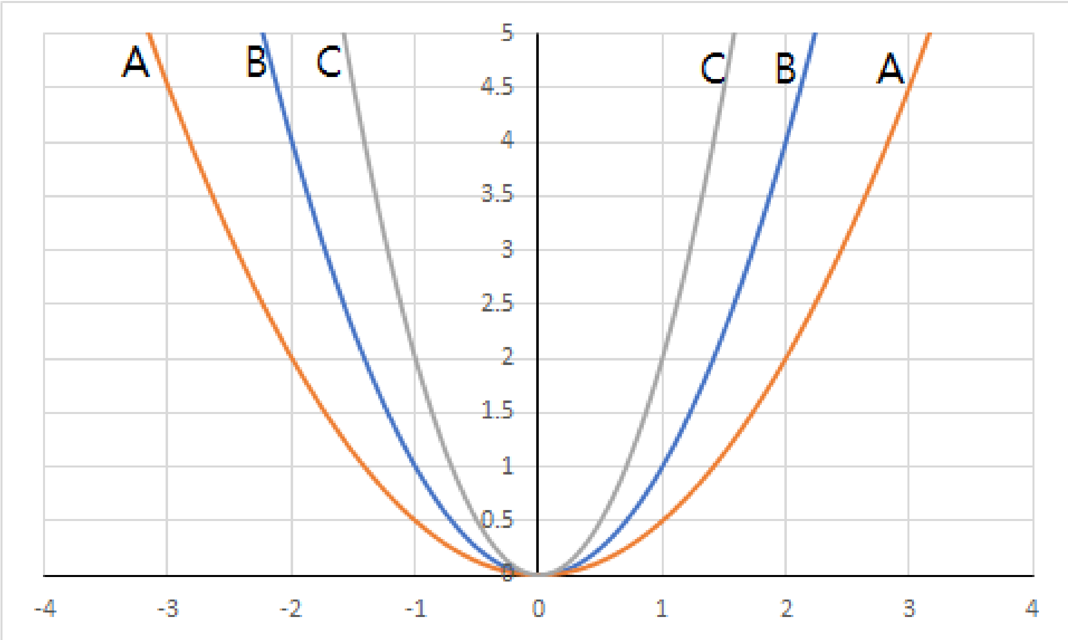

Figure 11: $y = x^2, 2x^2, \frac{x^2}{2}$

Summary

- In Cartesian coordinate, x-axis is horizontal, while y-axis vertical.
- The x-axis meets the y-axis at O, which is called the "origin," and is where both of the labeled numbers on two axes are zero.
- The slope is defined by the ratio of the y increase to the x increase.
- The slope of the graph of form $y = mx$ is m.
- When y divided by x is constant, we say y is "proportional" to x.
- The slope of the graph of the for $y = mx + b$ is m.
- We can describe a three dimensional space by using three dimensional Cartesian coordinate; in addition x-axis and y-axis, we add z axis.