Curl and Green's theorem

Suppose you calculate the line integral of $\vec{U} = U_x\hat{i} + U_y\hat{j}$ on the very small rectangular loop drawn in Fig.1. What is the value? We have:

$$
\begin{aligned}
\oint U \cdot d\vec{s} &= \int_{x_0}^{x_0+\Delta x} (U_x(x,y_0)\hat{i} + U_y(x,y_0)\hat{j}) \cdot dx\hat{i} + \int_{y_0}^{y_0+\Delta y} (U_x(x_0+\Delta x, y)\hat{i} + U_y(x_0+\Delta x, y)\hat{j}) \cdot dy\hat{j} \\
&+ \int_{x_0+\Delta x}^{x_0} (U_x(x, y_0+\Delta y)\hat{i} + U_y(x, y_0+\Delta y)\hat{j}) \cdot dx\hat{i} + \int_{y_0+\Delta y}^{y_0} (U_x(x_0,y)\hat{i} + U_y(x_0,y)\hat{j}) \cdot dy\hat{j} \\
&= \int_{x_0}^{x_0+\Delta x} (U_x(x,y_0) - U_x(x,y_0+\Delta y))\, dx + \int_{y_0}^{y_0+\Delta y} (U_y(x_0+\Delta x, y) - U_y(x_0,y))\, dy \\
&\approx \Delta x \left(-\frac{\partial U_x}{\partial y}\Delta y\right) + \Delta y \left(\frac{\partial U_y}{\partial x}\Delta x\right) = \Delta x \Delta y \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) \qquad (1)
\end{aligned}
$$

where $\oint$ means that the integration is over a closed loop. Therefore, we expressed the line integral of very small loop in terms of the area of the loop and the partial derivatives of the vector field concerned. Now, can we actually do better? Can we express a line integral of a big loop by using the same trick as well? To this end, let's consider Fig. 2. We have two very small loops. The line integral along the left loop is given by:

$$\left(\int_{\text{path1}} + \int_{\text{path2}} + \int_{\text{path3}} + \int_{\text{path4}}\right) \vec{U} \cdot d\vec{s} \qquad (2)$$

Similarly, the line integral along the right loop is given by:

$$\left(\int_{\text{path5}} + \int_{\text{path6}} + \int_{\text{path7}} - \int_{\text{path2}}\right) \vec{U} \cdot d\vec{s} \qquad (3)$$

Therefore, their sum is given by:

$$\left(\int_{\text{path1}} + \int_{\text{path3}} + \int_{\text{path4}} + \int_{\text{path5}} + \int_{\text{path6}} + \int_{\text{path7}}\right) \vec{U} \cdot d\vec{s} \qquad (4)$$

However, this is equal to the line integral along the rectangular loop composed by the addition of two smaller loops. See Fig.3. In other words, the above integral is equal to the following:

$$\left(\int_{\text{path1}} + \int_{\text{path5}} + \int_{\text{path6}} + \int_{\text{path7}} + \int_{\text{path3}} + \int_{\text{path4}}\right) \vec{U} \cdot d\vec{s} \qquad (5)$$

Therefore, we can obtain the line integral along the big loop if we add the line integral along the two smaller loop. Similarly, we can obtain line integral of a big loop if we add up the line integral along small loops which compose the big loop. See Fig. 4. Actually, one can also show that the big loop doesn't have to be the shape of zig-zag. If the very small loops that compose the big loop are really small, one can approximate the region the big loop encloses by very small loops really well. See Fig.5. Therefore, the line integral along the big loop is the sum (i.e. integration) of the last line of (1). Therefore, we can write:

$$\oint \vec{U} \cdot d\vec{s} = \int \int \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \right) dA \tag{6}$$

where dA denotes $dx\,dy$ i.e. the area.

We have actually calculated this in two dimensions; there were x and ys but no zs. But, actually, one can do this in three dimensions, and the result is following:

$$\oint_{\partial\Omega} \vec{U} \cdot d\vec{s} = \int_{\Omega} (\nabla \times U) \cdot dA \tag{7}$$

where $\partial\Omega$ is boundary of Ω and where we have the following definition of "curl":

$$\text{curl}\vec{U} \equiv \nabla \times \vec{U} \equiv \left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} \right) \hat{k} + \left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \right) \hat{j} + \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \right) \hat{k} \tag{8}$$

In our two dimensional example, we had only the last term, the one multiplied by $\hat{k}$, and dA was given by $dx\,dy\,\hat{k}$ which, when dot-producted with the former, yields (6).

Now, the interpretation of curl. Consider again the small rectangular loop in Fig.1. Remember that U_y didn't contribute to the integration over dx while U_x didn't contribute to the integration over dy. This allows us to consider only U_x when integrating over dx and U_y when integrating over dy. See Fig.6. I drew a very small pinwheel, instead of the very small rectangle as in Fig.1, and the relevant components of $\vec{U}$ that contribute to the line integral. Let's think of U as a wind. Then, notice that the pinwheel have more tendency to rotate anti-clockwise, the bigger $\big(U_x(x, y_0) - U_(x, y_0 + \Delta y)\big)$ and the bigger $(U_y(x_0 + \Delta x, y) - U_y(x_0, y))$. However, using derivatives, this is exactly given by the right-hand side of (6). Therefore, the "curl" really deserves its name. It indicates how much the vector field "rotate." The curl of the vector field is perpendicular to the rotating plane of the vector. For example, if the vector field rotates in the direction inside $x - y$ plane as in our example, then its curl is aligned along z direction. Similarly, if the vector field rotates in the $y - z$ plane, its curl is aligned along x-direction. If the rotating direction is reversed, for example, from anti-clockwise to clockwise, then the curl's direction is also reversed.

Final remark. The "loop" in loop quantum gravity is similar to loop considered in this article. It is not the loop in Feynman diagram.

Problem 1. Show that the curl of a conservative force always vanishes. Likewise, show that if the curl of a force vanishes it's conservative.(Hint[1])

Summary

- "Curl" tells you how much a vector field rotates. If it rotates in $x - y$ plane, $z - x$ plane, and $y - z$ plane respectively, its curl lies in z direction, and y direction and x direction respectively.

[1] If you can't figure it out, you can cheat by reading the next article

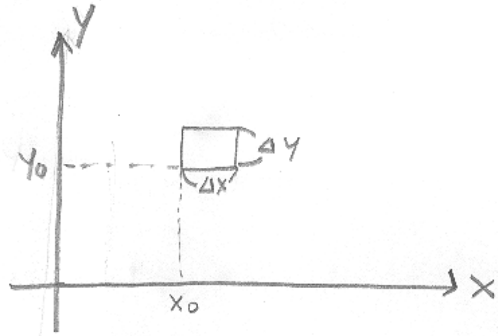

Fig. 1

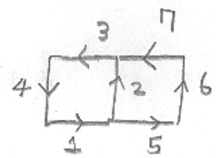

Fig. 2

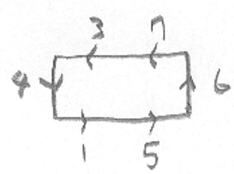

Fig. 3

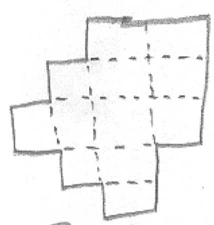

Fig. 4

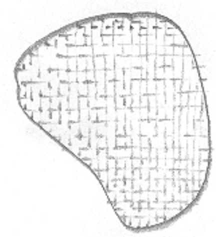

Fig. 5

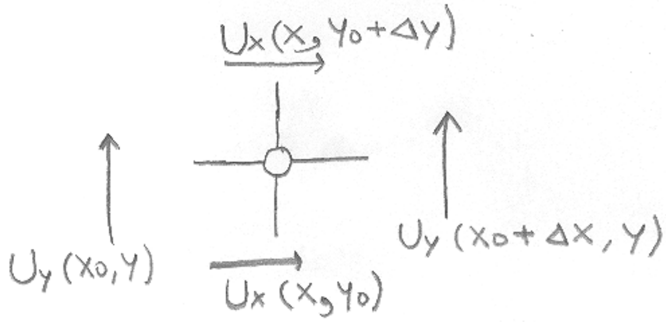

Fig. 6

- Green's theorem is given by

$$\oint_{\partial\Omega} \vec{U} \cdot d\vec{s} = \int_{\Omega} (\nabla \times U) \cdot dA$$