Why is the curl of gradient always zero?

The curl of gradient of any function is always zero. One can check this by using the definition of curl and gradient as follows:

$$\nabla \times \nabla f = \nabla \times (\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}) \tag{1}$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right)\hat{i} + \cdots \tag{2}$$

However, this is not a coincidence, as everything happens for reasons. Consider the following integration around some area Ω. Using Green's theorem, we have:

$$\int\int_{\Omega} (\nabla \times \nabla f) \cdot dA = \oint_{\partial\Omega} \vec{\nabla} U \cdot d\vec{s} = 0 \tag{3}$$

If you don't understand the last step that leads to zero, please check out (16) and (17) of our earlier article "Kinetic energy and Potential energy in three dimensions, Line integrals and Gradient."

Similarly, it turns out that the divergence of the curl is also zero. One can check this by explicit calculations. Naturally, it also happens for some reasons. Applying Stoke's theorem, and Green's theorem successively, we have:

$$\int\int\int_{\Omega} (\nabla \cdot (\nabla \times \vec{U})) \cdot dV = \int\int_{\partial\Omega} (\nabla \times \vec{U}) \cdot dA = \int_{\partial^2\Omega} \vec{U} \cdot d\vec{s} \tag{4}$$

However, $\partial^2\Omega$, the boundary of boundary of three-volume is always zero. For example, Earth's boundary is the surface of Earth where people live in and there is no boundary to the surface of Earth, even though ancient people falsely believed that there is an end to Earth.

In our article on differential forms, we will generalize not only Stoke's theorem and Green's theorem, but also the consideration we had in the present article, namely curl of gradient and divergence of curl being zero.

Summary

- The curl of a gradient is always zero.
- The divergence of a curl is always zero.