Faraday's law of induction in Maxwell's equations

In an earlier article, I have mentioned that an electric current in a loop is induced if the amount of magnetic flux through the loop changes. In this article, we will express this in terms of mathematical formulas.

First, magnetic flux can be expressed as follows:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{1}$$

Second, notice that the line integral of electric field along the loop is not zero, when Φ_B changes. We can see this from following reasoning. The line integral of electric field is exactly electric potential. Therefore, if the line integral along the loop were zero, the electric potential would be well-defined, (as the electric force would be a conservative force) which in turn implies electric current around the loop would not have been induced, since the electric charge concerned would fall into the lowest potential energy in the loop (i.e. the lowest electric potential for the positive charge and the highest electric potential for the negative charge) and stop there, rather than gaining energy to rotate constantly around the loop as in our case of changing magnetic flux. As the more electric field is induced the more magnetic flux changes, we can see that the line integral of the electric field is proportional to the time derivatives of Φ_B. This leads to the following equation:

$$\oint \vec{E} \cdot d\vec{s} = -k\frac{d\Phi_B}{dt} \tag{2}$$

where k is some proportionality constant and the minus sign indicates that the electric current is induced that opposes the change of magnetic flux as in Lenz's law. In our later article on "Faraday's law of induction and the Lorentz force" we will explain why the point of view that Faraday's law of induction can be partially explained from the Lorentz force enforces the value of k to be 1. (We explained this view qualitatively in our earlier article "Faraday's law of induction from the point of view of magnetic force on moving charge.") Plugging $k = 1$, we conclude:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{3}$$

where $\mathcal{E}$ is induced electromotive force, which denotes the "gained" electric potential upon rotating the circuit once. Let's re-express the above equation using curl. Using Green's theorem and (1), (3) can be re-expressed as:

$$\int_\Sigma (\nabla \times \vec{E}) \cdot d\vec{A} = -\frac{d\int_\Sigma \vec{B} \cdot d\vec{A}}{dt} \tag{4}$$

where Σ denotes the integration range. Now, notice

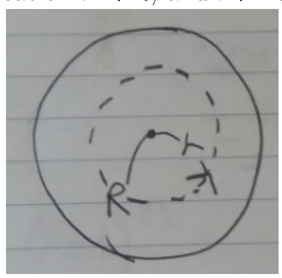
$$\frac{d \int_\Sigma \vec{B} \cdot d\vec{A}}{dt} = \int_\Sigma \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} + \int_{d\Sigma/dt} \vec{B} \cdot d\vec{A} \tag{5}$$

If we consider Σ that doesn't change over time, the last term on the right-hand side is 0. Thus, (4) implies

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

This is Faraday's law of induction expressed using curl.

Problem 1. See the figure below. In the circular region with radius R, there is a uniform magnetic field going into the page which increases at the rate $\dot{B}$ per unit time. Find the induced electric field at the location $r < R$, and $r > R$. (Hint[1])



Summary

- $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ i.e.

  $$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

  where Φ_B is the magnetic flux passing the loop.

[1]This problem is similar to Problem 3 in "Ampere's law."