Fermi energy

In the last article, we have seen that a significant number of bosons occupy the ground state (i.e. state with $\epsilon = 0$) at very low temperature. However, the same cannot be true for fermions because of Pauli's exclusion principle; if the ground state is already filled in by certain fermions, other fermions of the same kind cannot occupy the ground state, and must occupy the next lowest energy states. If you put more fermions of the same kind, they must occupy the third lowest energy states and so on.

Suppose we put N fermions in volume V at temperature zero. Then, up to which energy are the fermions filled in? That is called "Fermi energy," and we will obtain it in this article.

Recall that average number of particles per a given state with energy ϵ is given by the Fermi-Dirac distribution as follows:

$$\langle n \rangle = \frac{1}{e^{(\epsilon-\mu)/kT} + 1} \tag{1}$$

It is easy to check that for $T = 0$, $n = 1$ for $\epsilon < \mu$, and $n = 0$ for $\epsilon > \mu$. It means that at zero temperature the states are completely filled in for $\epsilon < \mu$, and no states with $\epsilon > \mu$ are filled in.

When we consider a system of fermions, we usually consider fermions with spin 1/2 such as electrons, protons and neutrons. For these fermions, there are two states with the same energy. Spin up and spin down. So, the degeneracy is 2. Therefore, we have

$$N = \frac{2}{h^3} \int \frac{d^3p d^3q}{e^{(\epsilon-\mu)/kT} + 1} \tag{2}$$

If the fermion can be treated non-relativistically, we have $p = \sqrt{2m\epsilon}$. As $d^3p = 4\pi p^2\, dp$, the above formula becomes

$$\frac{N}{V} = \frac{4\pi(2m)^{3/2}}{h^3} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon-\mu)/kT} + 1} \tag{3}$$

When $T = 0$, it becomes

$$\frac{N}{V} = \frac{4\pi(2m)^{3/2}}{h^3} \int_0^\mu \epsilon^{1/2} d\epsilon \tag{4}$$

Thus, we get

$$\mu = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{5}$$

This is the Fermi energy, which is usually denoted by E_F. Fermi momentum is given by the momentum a fermion has when its (kinetic) energy is the Fermi energy. Fermi velocity is defined similarly. Fermi temperature is defined by

$$T_F = \frac{E_F}{k} \tag{6}$$

Notice that the occupancy number of fermions deviate significantly from the one at zero temperature, only when T is not negligible compared to T_F. In other words, only when the temperature is comparable to the Fermi temperature, do the fermions move significantly faster than at zero temperature. The fermi temperature for electrons in a metal is roughly hundred to thousand times bigger than room temperature. In other words, at room temperature, the free electrons do not move significantly faster than Fermi velocity. (Free electrons are the electrons which are not bounded by particular atoms, but move freely in a metal.)

According to Wikipedia the Fermi energy is an important concept in the solid state physics of metals and superconductors, the physics of quantum liquids like low temperature helium, nuclear physics and to understanding the stability of white dwarf stars against gravitational collapse. But, I am not an expert, so I stop here.

Summary

- Bose-Einstein condensation cannot happen for fermions because of Pauli's exclusion principle.
- At temperature zero, the states are filled by fermions up to Fermi energy.