What is a Feynman diagram?

In earlier articles, I explained what the expectation values in quantum field theory mean. Just to recap, and to introduce a more systematic notation, let me rewrite the formula here.

$$\langle f\left(x_1, x_2, x_3, \ldots x_n\right)\rangle = \frac{\int_{-\infty}^{\infty} f(x_1, x_2 \ldots x_n)\exp[-S\left(x_1, x_2 \ldots x_n\right)]d^n x}{\int_{-\infty}^{\infty} \exp[-S\left(x_1, x_2 \ldots x_n\right)]d^n x}$$

Here, both f and S are assumed to be polynomials. If they aren't, one can always make them polynomials by using Taylor series and ignore higher order terms. Of course, one must assume the convergence of the Taylor series, and the sum of order by order expansions on the right-hand side will also converge, but for the present purpose, let's assume that they do. In such a case, physicists say that "the perturbative expansions are valid."

For simplicity, let's also assume that the terms linear in x are absent from S. One can always make it so by redefining the variable x.

If S is quadratic in x, we call such a theory "free." If S has higher order terms in x, we say that such a theory has interactions.

Now, I will explain what Feynman diagram is. Let us consider the simpler case of a free theory first, before delving into the interacting theory. Let $f\left(x_1, x_2, x_3, \ldots x_n\right) = x_1 x_2$, and $S\left(x_1, x_2 \ldots x_n\right) = \frac{A_{ij}}{2} x^i x^j$. Then, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} x_1 x_2 e^{-\frac{A_{ij}}{2}x^i x^j + B_i x^i} d^n x &= \sqrt{\frac{(2\pi)^n}{\det A}} \frac{\partial^2 e^{\frac{1}{2}B^T A^{-1} B}}{\partial B_1 \partial B_2} = \sqrt{\frac{(2\pi)^n}{\det A}} \frac{\partial[\left(A^{-1}\right)^{1j} B_j \ e^{\frac{1}{2}B^T A^{-1} B}]}{\partial B_2} \\ &= \sqrt{\frac{(2\pi)^n}{\det A}} [\left(A^{-1}\right)^{12} + \left(A^{-1}\right)^{1j} B_j \ \left(A^{-1}\right)^{2k} B_k \ ] e^{\frac{1}{2}B^T A^{-1} B} \qquad (1)\end{aligned}$$

Here, $\left(A^{-1}\right)^{jk}$ denotes $(j,\ k)$ components of the inverse matrix of A. By setting Bs equal to zero, and dividing this expression by $\int_{-\infty}^{\infty} e^{-\frac{A_{ij}}{2}x^i x^j} d^n x$, we obtain the following:

$$\langle x_1 x_2 \rangle = \left(A^{-1}\right)^{12}$$

We denote this value by the following Feynman diagram:

Fig.1

The line denotes A^{-1} and the numbers indicate that they are (1, 2) component of this matrix. Such prescriptions for the value of Feynman diagrams are called “Feynman rules.”

Now, let’s consider a slightly more complicated case. Let $f(x_1, x_2, x_3, \ldots x_n) = x_1x_2x_3x_4$ Then we get:

$$\langle x_1x_2x_3x_4\rangle = \left(A^{-1}\right)^{12}\left(A^{-1}\right)^{34} + \left(A^{-1}\right)^{13}\left(A^{-1}\right)^{24} + \left(A^{-1}\right)^{14}\left(A^{-1}\right)^{23}$$

The proof is left as an exercise to the readers. I strongly encourage you to do this calculation so that you can see how it works. Feynman diagrams for this expectation value are following.

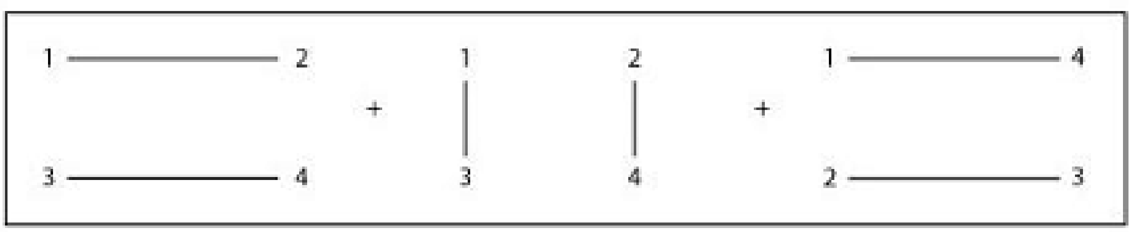


Fig.2

The basic idea is that you choose all possible contractions; that is you connect pairs of points in all possible ways such that each point lies on exactly one line. One can pictorially represent this as follows:

$$\begin{aligned}\langle x_1x_2x_3x_4\rangle &= x_1x_2x_3x_4 + x_1x_2x_3x_4 + x_1x_2x_3x_4 \\ &= \langle x_1x_2\rangle\langle x_3x_4\rangle + \langle x_1x_3\rangle\langle x_2x_4\rangle + \langle x_1x_4\rangle\langle x_2x_3\rangle \qquad (2)\end{aligned}$$

By similar reasoning, we have:

$$\langle x_1x_2x_3\rangle = 0$$

Since there are an odd number of points (1, 2, 3), in all possible contractions, one point is always left out. Pictorially, we can represent this as the following.

$$\langle x_1x_2x_3\rangle = \langle x_1x_2x_3\rangle + \langle x_1x_2x_3\rangle + \langle x_1x_2x_3\rangle = 0$$

So, this expectation value has to be 0. For the same reason, the expectation value for an odd number of xs is always zero in a free theory.

Now, we consider the interacting case. Let

$$S = [-\sum_i \lambda_i {x_i}^4] + \frac{A^{ij}}{2} x_i x_j$$

For simplicity, we will consider the case that $\lambda_i = 0$ for $i = 1,\ 2,\ 3,\ 4$ and $\lambda_i = \lambda$ otherwise. Then, for the expectation value of four xs, which physicists call "four-point function," we have the following.

$$\langle x_1 x_2 x_3 x_4 \rangle_{int} = \frac{\int_{-\infty}^{\infty} x_1 x_2 x_3 x_4 \exp\left[\{\sum_i \lambda_i {x_i}^4\} - \frac{A^{ij}}{2} x_i x_j\right] d^n x}{\int_{-\infty}^{\infty} \exp\left[\{\sum_i \lambda_i {x_i}^4\} - \frac{A^{ij}}{2} x_i x_j\right] d^n x} \tag{3}$$

Here, "int" denotes that the theory is interacting. Let's calculate this expectation value up to the first order in λ, via a Taylor expansion:

$$\langle x_1 x_2 x_3 x_4 \rangle_{int} = \frac{\langle x_1 x_2 x_3 x_4 \rangle + \langle x_1 x_2 x_3 x_4 \sum_i \lambda_i {x_i}^4 \rangle + \dots}{1 + \langle \sum_i \lambda_i {x_i}^4 \rangle + \dots}$$

The first term in the numerator gives the result of free theory. The first term in the denominator is 1 and the second term in the denominator gives the following:

$$\sum_i \lambda_i \langle x_i^4 \rangle = \sum_i \lambda_i [x_i x_i x_i x_i + x_i x_i x_i x_i + x_i x_i x_i x_i] = \sum_i 3\lambda_i \langle x_i x_i \rangle^2$$

We can represent this by the following Feynman diagram:

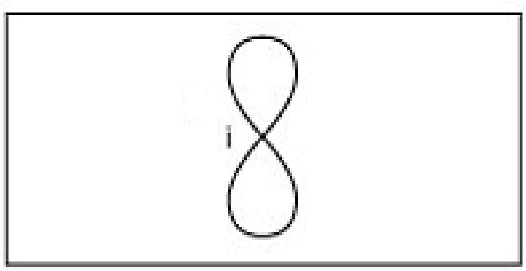


Fig.3

Notice that we can create another Feynman rule saying that a vertex with four out-going lines corresponds to a factor of λ. Actually, we have extra factor of 3 here, but let's ignore it as it is beyond the scope of this article, even though there is a way to count this extra factor from the Feynman diagram. Now, we calculate the second term of the numerator. We get:

$$\begin{aligned} \langle x_1 x_2 x_3 x_4 \sum_i \lambda_i {x_i}^4 \rangle &= \langle x_1 x_2 x_3 x_4 \rangle \langle \sum_i \lambda_i {x_i}^4 \rangle + \sum_i 4! \lambda_i \langle x_1 x_i \rangle \langle x_2 x_i \rangle \langle x_3 x_i \rangle \langle x_4 x_i \rangle \\ &+ \sum_i 12\lambda_i \langle x_i x_i \rangle \Big[\langle x_1 x_2 \rangle \langle x_3 x_i \rangle \langle x_4 x_i \rangle + \langle x_1 x_3 \rangle \langle x_2 x_i \rangle \langle x_4 x_i \rangle + \langle x_1 x_4 \rangle \langle x_2 x_i \rangle \langle x_3 x_i \rangle \\ &\langle x_2 x_3 \rangle \langle x_1 x_i \rangle \langle x_4 x_i \rangle + \langle x_2 x_4 \rangle \langle x_1 x_i \rangle \langle x_3 x_i \rangle + \langle x_3 x_4 \rangle \langle x_1 x_i \rangle \langle x_2 x_i \rangle \Big] \end{aligned} \tag{4}$$

The factor 4! comes out because there are 4! ways to contract x_1, x_2, x_3, x_4 with x_is. The factor 12 comes out because there are 4×3 ways to contract x_3, x_4 with x_is and so on. To express the above formula using a Feynman diagram, we write:

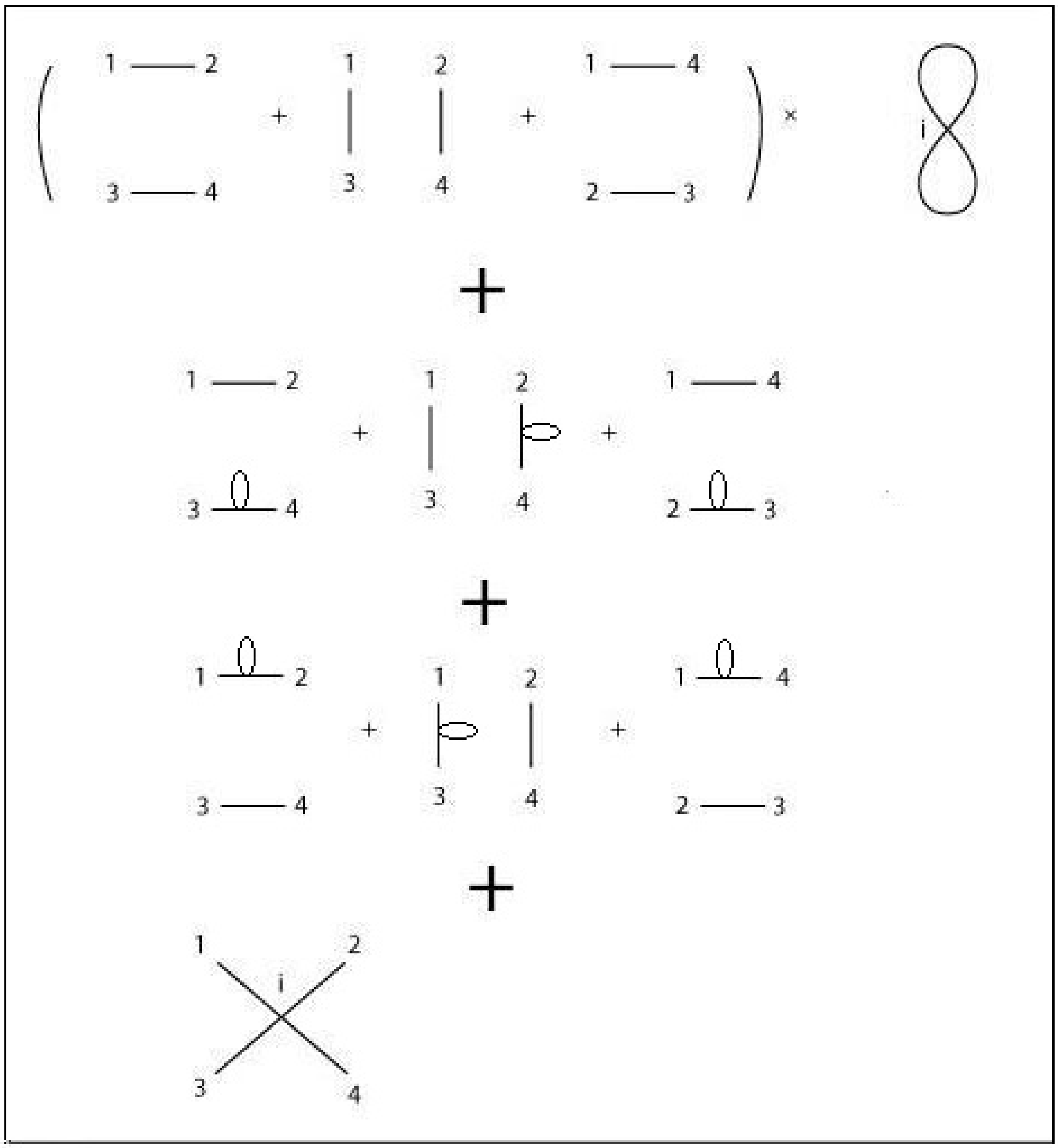


Fig.4

Finally, putting the denominator and numerator together, we get:

$$\langle x_1x_2x_3x_4\rangle_{int} = \langle x_1x_2x_3x_4\rangle + \sum_i 4!\lambda_i\langle x_1x_i\rangle\langle x_2x_i\rangle\langle x_3x_i\rangle\langle x_4x_i\rangle$$

$$+\sum_i 12\lambda_i\langle x_ix_i\rangle\Big[\langle x_1x_2\rangle\langle x_3x_i\rangle\langle x_4x_i\rangle + \langle x_1x_3\rangle\langle x_2x_i\rangle\langle x_4x_i\rangle + \langle x_1x_4\rangle\langle x_2x_i\rangle\langle x_3x_i\rangle$$

$$\langle x_2x_3\rangle\langle x_1x_i\rangle\langle x_4x_i\rangle + \langle x_2x_4\rangle\langle x_1x_i\rangle\langle x_3x_i\rangle + \langle x_3x_4\rangle\langle x_1x_i\rangle\langle x_2x_i\rangle\Big] + \mathcal{O}(\lambda^2) \qquad (5)$$

In Feynman diagrams, we can represent them as:

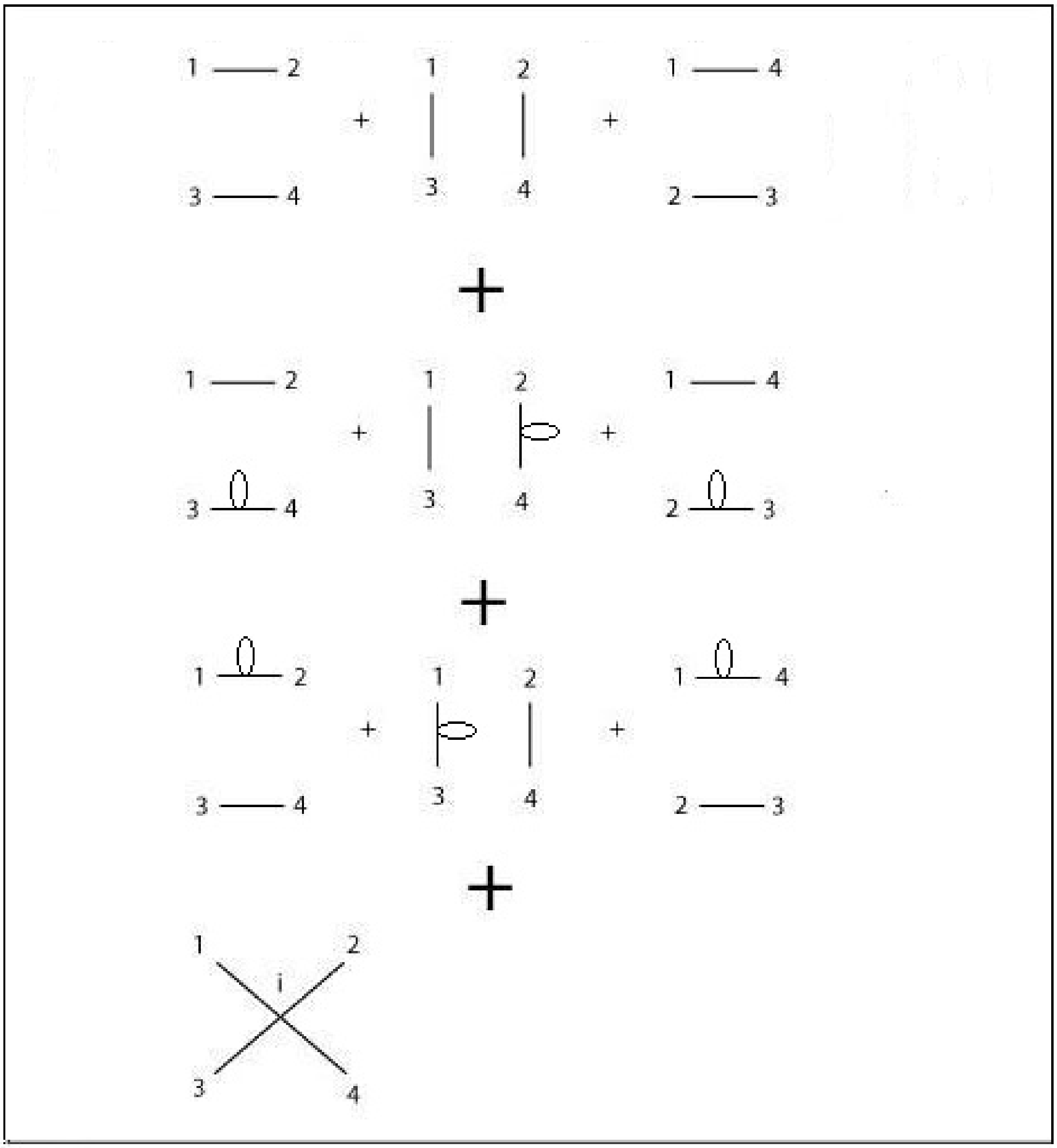
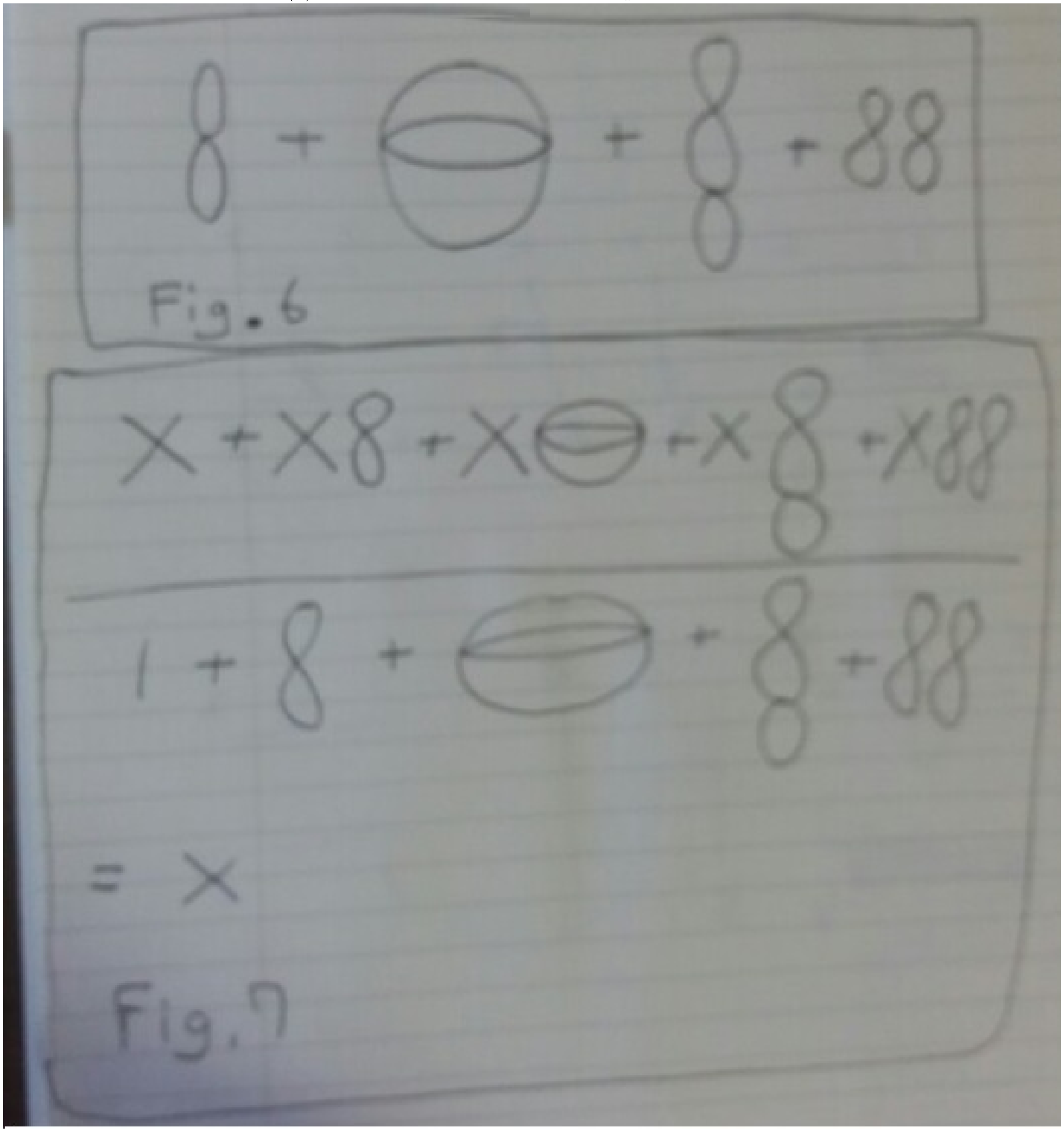


Fig.5

This is neat, as the awkward diagrams such as "8" are canceled out, and not present in the final Feynman diagrams. But, is this a coincidence? Why are they canceled out? Actually, it's how the denominator of (3) is destined to work. Let me explain this in more detail.



There are two kinds of points in the Feynman diagrams: external points and internal points. Each external point such as x_1, x_2, x_3 and x_4 is connected to only one other point while each internal point such as x_i is connected to more than one other points. As the denominator doesn't have any x_1, x_2, x_3, x_4, the corresponding Feynman diagrams don't have any external points but only have internal points. Possible Feynman diagrams are drawn in Fig.6. They are sometimes called "vacuum bubbles."

On the other hand, each Feynman diagram for the numerator has four external points. Each Feynman diagram is a product of (one or more) sub-diagrams, of which the number of the external points sums up to four. Therefore, some of these sub-diagrams can be vacuum bubbles, which have no external points. Therefore, the vacuum bubbles factor can be factored out and canceled out when divided by the denominator. Fig. 7 is the example.

So far, I have described the mathematical aspects of Feynman diagrams. Let me describe now their physical aspects. The two point function $\langle x_i x_j \rangle$, or equivalently, $\left(A^{-1}\right)^{ij}$ is called "the propagator." It is related to the amplitude that a particle located at the point x_i moves to x_j. Similarly, the four-point function is related to the amplitude that two particles move from two points to two other points. Notice that in the case of a free theory, like that of Fig. 2, these two particles do not interact each other, as the two lines never meet and is simply given by the sum of products of two propagators. However, if there is an interaction, the Feynman diagrams include the diagrams which include lines that meet each other such as "$\times$" diagram of Fig. 5.

Problem 1. Draw all possible types for Feynman diagrams for $\langle x_1 x_2 x_3 x_4 \rangle_{int}$ in λ^2 order. (i.e. you don't need to label the points by 1, 2, 3, 4, i and so on.)