The Jacobian and change of variables

As you may already know from your single-variable calculus course, change of variable is useful in evaluating integration. Same is true for multi-variable calculus. This is the topic of this article.

Suppose you want to calculate the following:

$$\int \int h(s,t) ds dt \tag{1}$$

Now, suppose you notice that you can integrate this easily if you use a set of other variables u, v. Let's say s and t are related with u and v by following relation:

$$s = s(u,v), \qquad t = t(u,v) \tag{2}$$

Given this, you may perhaps want to attempt to calculate the integration by replacing $h(s,t)$ by $h(s(u,v), t(u,v))$, and expressing ds and dt in terms of du and dv. The first part is straightforward, however the second part is baffling, since both ds and dt are functions of du and dv, as follows:

$$ds = \frac{\partial s}{\partial u} du + \frac{\partial s}{\partial v} dv \tag{3}$$

$$dt = \frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv \tag{4}$$

So, if we multiply ds and dt, we will have $dudu$, $dudv$, $dvdu$, and $dvdv$, simply not knowing how to take care of all these terms. Actually, this is a wrong way. Let's see the correct way.

For simplicity, let's consider the case $h = 1$ first, as we will be able to consider the general case later from this. Then, we have:

$$\int_D ds dt = \int_{D'} M du dv \tag{5}$$

where D' is the region in u and v variables that correspond to D, and M is a factor that needs to be determined. Now, consider the case in which D' is given by following:

$$\int_{D'} = \int_v^{v+\Delta v} \int_u^{u+\Delta u} \tag{6}$$

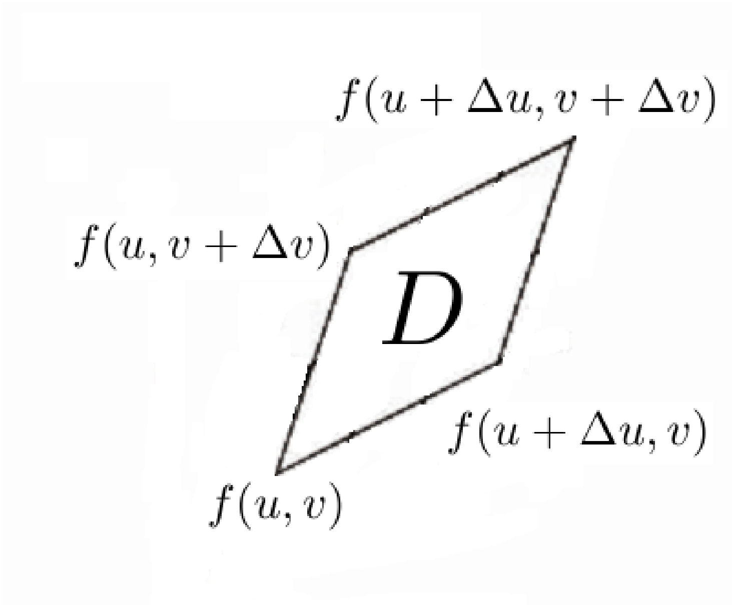

Figure 1: Change of variables

Then, we will have:

$$\text{Area of D} = M\Delta u\Delta v \tag{7}$$

Now, let's draw D to calculate its area. For simplicity, we will use the following notation:

$$f(u,v) = (s,t) \tag{8}$$

See Fig.1 Now, let's Taylor-expand f.

$$f(u+\Delta u, v) = (s + \frac{\partial s}{\partial u}\Delta u, t + \frac{\partial t}{\partial u}\Delta u) \tag{9}$$

$$f(u, v+\Delta v) = (s + \frac{\partial s}{\partial v}\Delta v, t + \frac{\partial t}{\partial v}\Delta v) \tag{10}$$

$$f(u+\Delta u, v+\Delta v) = (s + \frac{\partial s}{\partial u}\Delta u + \frac{\partial s}{\partial v}\Delta v, t + \frac{\partial t}{\partial u}\Delta u + \frac{\partial t}{\partial v}\Delta v) \tag{11}$$

Then, it is easy to see that the area of the parallelogram is given by following, if you remember our earlier article on determinant.

$$\text{Area of D} = \det \begin{pmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{pmatrix} \Delta u\Delta v \tag{12}$$

where det denotes determinant. Then, from (7,)we conclude

$$M = \det \begin{pmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{pmatrix} \tag{13}$$

The matrix in the above equation is called "Jacobian matrix," This implies

$$\int_D dsdt = \int_{D'} \frac{\partial(s,t)}{\partial(u.v)} dudv \tag{14}$$

where $\frac{\partial(s,t)}{\partial(u.v)}$ denotes the determinant of Jacobian matrix. In conclusion,

$$\int_D f(s,t)dsdt = \int_{D'} f(s(u,v),t(u,v)) \frac{\partial(s,t)}{\partial(u.v)} dudv \tag{15}$$

It goes without saying that this formula can be generalized to the cases when the number of variables for the integration is bigger.