Maxwell's equations in differential forms

In this article, I will show that Maxwell's equations can be re-expressed in a very simple form if we use differential forms. The electromagnetic field can be expressed as a two-form as follows:

$$F = (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \tag{1}$$

or equivalently,

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{2}$$

where E_x, E_y and E_z are the x, y and z components of the electric field, and B_x, B_y and B_z are the x, y, and z components of the magnetic field.

Similarly, a dual form $*F$ is given as follows:

$$*F = -E_x dy \wedge dz - E_y dz \wedge dx - E_z dx \wedge dy + (B_x dx + B_y dy + B_z dz) \wedge dt \tag{3}$$

(* is called "the Hodge star operator" and is described in our article "Vierbein formalism and Palatini action in general relativity.")

Finally, we can define the current density of charge as follows:

$$J = (j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy) \wedge dt - \rho dx \wedge dy \wedge dz \tag{4}$$

where j_x, j_y, and j_z are the x, y, and z components of the charge current density and ρ is the charge density.

With this notation, Maxwell's equations can be written as $dF = 0$ and $d * F = J$. One can show this by explicit calculation as follows:

$$\begin{aligned} dF &= \frac{\partial E_x}{\partial y} dy \wedge dx \wedge dt + \frac{\partial E_x}{\partial z} dz \wedge dx \wedge dt + \cdots \\ &= (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t}) dx \wedge dy \wedge dt + (\mathrm{div}\vec{B}) dx \wedge dy \wedge dz + \cdots \end{aligned} \tag{5}$$

Then $dF = 0$ leads to

$$\mathrm{curl}\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{6}$$

$$\mathrm{div}\vec{B} = 0 \tag{7}$$

or equivalently,

$$dF = \frac{1}{2}\partial_\lambda F_{\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu = \frac{1}{2}\partial_\nu F_{\lambda\mu} dx^\nu \wedge dx^\lambda \wedge dx^\mu = \frac{1}{2}\partial_\mu F_{\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0$$

$$
\begin{aligned}
&= \frac{1}{3}dF + \frac{1}{3}dF + \frac{1}{3}dF = 0 \\
&= \frac{1}{6}(\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda})dx^\lambda \wedge dx^\mu \wedge dx^\nu = 0 \\
&\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0
\end{aligned} \tag{8}
$$

Here we see that the Bianchi identity, introduced in our earlier article "Non-Abelian gauge theory," can be used to express half of Maxwell's equations. Similarly $d * F = j$ leads to

$$\mathrm{curl}\vec{B} - \frac{\partial \vec{E}}{\partial t} = j \tag{9}$$

$$\mathrm{div}\vec{E} = \rho \tag{10}$$

or equivalently,

$$\partial^\mu F_{\mu\nu} = J_\nu \tag{11}$$

where ∂^μ=$(\partial^0, \partial^1, \partial^2, \partial^3)$ is $(-\partial_0, \partial_1, \partial_2, \partial_3)$.

Notice that $dJ = 0$, since $d(d{*}F) = 0$ follows from $d^2 = 0$. $dJ = 0$ implies the conservation of charge. i.e.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{j} = 0 \tag{12}$$

Of course, that Maxwell's equations imply the conservation of charge is well-known, but this result is much easier to see in the differential form formalism. Another important point is that the mathematical formula expressing the electromagnetic fields in terms of the electric potential and the vector potential can also be expressed succinctly in terms of differential forms as follows:

$$F = dA \tag{13}$$

where A represents the combination of the electric potential and the magnetic potential into a single electromagnetic potential as

$$A = -\phi dt + A_x dx + A_y dy + A_z dz \tag{14}$$

or equivalently

$$A = A_\nu dx^\nu \tag{15}$$

Using explicit calculation once again, one obtains the following:

$$E = -\mathrm{grad}\phi - \frac{\partial \vec{A}}{\partial t} \tag{16}$$

$$B = \mathrm{curl}\vec{A} \tag{17}$$

or equivalently,

$$
\begin{aligned}
F = dA &= \partial_\mu A_\nu dx^\mu \wedge dx^\nu = \partial_\nu A_\mu dx^\nu \wedge dx^\mu = -\partial_\nu A_\mu dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}dA + \frac{1}{2}dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned} \tag{18}
$$

Notice also that, for a given electromagnetic field F, the electromagnetic potential A is not uniquely determined. Indeed, the corresponding electromagnetic field of the following electromagnetic potential

$$A' = A + d\theta \tag{19}$$

is the same as the corresponding electromagnetic field of A. (Here, θ is an arbitrary 0-form, i.e., a function.) One can easily see this by direct calculation:

$$dA' = d(A + d\theta) = dA + dd\theta = dA \tag{20}$$

The above formula relating A' with A is called a "gauge transformation." In this case, the electromagnetic potential is also called the "gauge potential," and "choosing a gauge" means choosing A among all possible As for a given F. Here, performing gauge transformation implies choosing another gauge.

Maxwell's equations are generalized in string theory by using the notation of differential forms. In string theory, where spacetime is 10-dimensional, the degrees of the electric field and the magnetic field don't need to be the same and don't need to be two, as they are in four dimensions; depending on the theory, the degrees of the electric and magnetic fields can take any values from 1 to 10, provided that together they add up to 10, which is the number of spacetime dimensions in string theory. Notice that in the 4-dimensional case, the degrees of the electric field and the magnetic field add up to 4, the number of spacetime dimensions in our daily lives.