

The Poisson bracket

The Poisson bracket of two quantities f and g is defined as follows:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \quad (1)$$

One of the properties of Poisson bracket is that it is anti-symmetric. In other words,

$$\{f, g\} = -\{g, f\} \quad (2)$$

It is left as an exercise to readers to verify this, as it is obvious from the definition of the Poisson bracket. This also implies:

$$\{f, f\} = -\{f, f\} \quad (3)$$

which, in turn, implies:

$$\{f, f\} = 0 \quad (4)$$

Now, let's calculate the following quantity:

$$\{f, H\} = \sum_i \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} \quad (5)$$

$$= \sum_i \frac{\partial f}{\partial q^i} \dot{q}^i + \frac{\partial f}{\partial p_i} \dot{p}_i \quad (6)$$

$$= \frac{df}{dt} \quad (7)$$

$$\{f, H\} = \frac{df}{dt} \quad (8)$$

Here, we see that Hamiltonian is closely related to the “time translation,” as physicists call it. ($f \rightarrow f + \frac{df}{dt} \delta t$).

Now, as advertised, let me explain why Hamiltonian is energy. Energy is something that is conserved, and remains constant as time goes. We can verify this using (4) and (8) as follows:

$$0 = \{H, H\} = \frac{dH}{dt} \quad (9)$$

There are a couple of more crucial properties of Poisson bracket. As an exercise, the readers should verify the following:

$$\{q^i, p_j\} = \delta_j^i \quad (10)$$

and, the following:

$$\{A, B + C\} = \{A, B\} + \{A, C\} \quad (11)$$

$$\{A, BC\} = B\{A, C\} + \{A, B\}C \quad (12)$$

which can be derived straightforwardly from the definition of Poisson bracket using Leibniz's rule. (i.e. $(fg)' = f'g + fg'$)

Problem 1. Prove the following formula.

$$f(t) = f_0 + \{f, H\}_0 t + \frac{t^2}{2!} \{\{f, H\}, H\}_0 + \frac{t^3}{3!} \{\{\{f, H\}, H\}, H\}_0 + \dots \quad (13)$$

where the subscript 0 denotes the value evaluated when $t = 0$.

Problem 2. Let's denote the position of an object in Cartesian coordinate by $x = q^1, y = q^2, z = q^3$ and its momentum by $p_x = p_1, p_y = p_2, p_z = p_3$. Then (10) implies $\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1$ and all other Poisson brackets vanish. Using this relation, along with $L_z = xp_y - yp_x$ (i.e. the z th component of angular momentum), calculate the following. The answer to this problem will turn out to be useful in solving a problem in our later article "Noether's theorem." (Hint¹)

$$\{x, L_z\} = ?, \quad \{y, L_z\} = ?, \quad \{z, L_z\} = ? \quad (14)$$

Summary

- $\{f, g\} = \sum_i \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}$
- $\{f, g\} = -\{g, f\}$ which implies $\{f, f\} = 0$.
- $\{f, H\} = \frac{df}{dt}$
- The Hamiltonian is conserved as $\{H, H\} = \frac{dH}{dt} = 0$.
- $\{q^i, p_j\} = \delta_j^i$.
- $\{A, B + C\} = \{A, B\} + \{A, C\}$
- $\{A, BC\} = B\{A, C\} + \{A, B\}C$

¹Use (11) and (12).