Poynting vector

In this article, we will show that electromagnetic field not only carries energy as we have seen in our last article, but it can also carry momentum.

Now, consider a region V. The energy due to electric field and magnetic field in this region is given by

$$\int_V u dV \tag{1}$$

where u is

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \tag{2}$$

However, the energy can flow in to the region V or out of the region V. So, we need to consider "energy flux," which is denoted as $\vec{S}$ and called "Poynting vector."

We naturally have

$$\int_V \frac{\partial u}{\partial t} = -\oint \vec{S}\cdot d\vec{A} = -\int \nabla\cdot\vec{S} dV \tag{3}$$

This equation is easy to understand, if you review what we have learned in previous articles. For continuity equation, we had

$$\int_V \frac{\partial \rho}{\partial t} = -\oint \vec{j}\cdot d\vec{A} = -\int \nabla\cdot\vec{j} dV \tag{4}$$

However, (3) is not entirely correct. Electromagnetic field does work on electric charge. The decrease of energy of electromagnetic field should take account this. So, let's consider the work done on electric charge.

To this end, first consider the Lorentz force on an electric charge q.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B} \tag{5}$$

If you consider the work done on the electric charge q per unit time, we have

$$\vec{F}\cdot\frac{d\vec{s}}{dt} = \vec{F}\cdot\vec{v} = q\vec{E}\cdot\vec{v} + q\vec{v}\cdot(\vec{v}\times\vec{B}) \tag{6}$$

As the second term is zero (i.e. magnetic field never does work), upon replacing the charge by the charge density integrated by volume, the above expression becomes

$$\vec{F}\cdot\frac{d\vec{s}}{dt} = \int \rho\vec{E}\cdot\vec{v} dV \tag{7}$$

Using $\vec{J} = \rho\vec{v}$,

$$\vec{F}\cdot\frac{d\vec{s}}{dt} = \int \vec{E}\cdot\vec{J} dV \tag{8}$$

Since the energy of electromagnetic field must decrease by this amount, instead of (3), we have

$$\int_V \frac{\partial u}{\partial t} = -\int \nabla \cdot \vec{S} dV - \int \vec{E} \cdot \vec{J} dV \tag{9}$$

Thus, we conclude

$$\frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = -\nabla \cdot \vec{S} \tag{10}$$

Plugging (2) to the above relation, we get

$$-\nabla \cdot \vec{S} = \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{\vec{B}}{\mu_0} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \vec{J} \tag{11}$$

Using the following Maxwell's equations

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{12}$$

$$\nabla \times \frac{\vec{B}}{\mu_0} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \tag{13}$$

we get

$$-\nabla \cdot \vec{S} = \frac{1}{\mu_0} \vec{E} \cdot (\nabla \times \vec{B}) - \frac{1}{\mu_0} \vec{B} \cdot (\nabla \times \vec{E}) \tag{14}$$

Then, one can check (**Problem 1.**)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \tag{15}$$

So, this means that electromagnetic field carries energy flux.

Now, recall what we have learned in "Light as electromagnetic waves." If light is moving in say z-direction, the electromagnetic fields that consist light lie in $x-y$ plane, and the electric field and magnetic field are orthogonal. Say, the electric field lies in x-direction (or y direction)then we know that the magnetic field lies in y-direction (or $-x$ direction) from Problem 3 of that article. Now plug in this direction to (15), then you get z-direction. Indeed, light is carrying energy flux in its moving direction!

The next question is, how much momentum does it carry given $\vec{S}$? Recall

$$\vec{S} = \frac{\text{energy}}{\text{area} \cdot \text{time}} = \frac{\text{energy}}{\text{volume}} \cdot \text{velocity} \tag{16}$$

We know that the velocity of light is c. We also know that light obeys energy=momentum$\cdot c$. Thus,

$$\vec{S} = \frac{\text{momentum} \cdot c}{\text{volume}} \cdot c \tag{17}$$

Thus, the momentum density $\vec{g}$ is given by

$$\vec{g} = \frac{\text{momentum}}{\text{volume}} = \frac{\vec{S}}{c^2} \tag{18}$$

Thus, we conclude that electromagnetic field carries momentum, and its density is given by above formula.

Now, we are in a position to answer the questions we raised in our earlier articles. Electromagnetic field carries momentum, and consequently, angular momentum. If you take into account the momentum and the angular momentum of the charges only, they do not seem to be conserved. But, if you take into account the momentum and the angular momentum of electromagnetic fields as well, the total momentum and angular momentum are conserved.

Let me conclude this article with two historical comments. First, Notice that in deriving the momentum of Poynting vector, we used the relation $E = pc$ for light. This relation was known before the special theory of relativity was discovered. It was actually derived by Maxwell by considering the change of momentum of electromagnetic waves upon being reflected on the surface of conductor (i.e. metal). Of course, this value would be different by factor of 2 if light were a non-relativistic particle that obeyed $E = pc/2$. Thus, Einstein's theory of relativity re-confirmed the known relation $E = pc$.

Second, Poynting vector was first derived by John Poynting, an English physicist in 1884. His advisor was James Maxwell. Notice that Poynting vector is not spelled as "Pointing vector," because it was named after the discoverer's name, even though Poynting vector is a vector "pointing" in which direction energy flows.

Summary

- Poynting vector tells you in which direction the energy of the electromagnetic field flows. It is proportional to $\vec{E} \times \vec{B}$.