Angular momentum in quantum mechanics

In this article, we will quantize angular momentum and calculate its eigenvalues. From classical mechanics, you may recall:

$$\vec{L} = \vec{r} \times \vec{p} \tag{1}$$

In other words, in Cartesian coordinate:

$$L_x = yp_z - zp_y \tag{2}$$

$$L_y = zp_x - xp_z \tag{3}$$

$$L_z = xp_y - yp_x \tag{4}$$

These formulas also make sense in quantum mechanics, if we interpret $L_x, L_y, L_z, x, y, z, p_x, p_y, p_z$ not as numbers but as operators (i.e. matrices).

It is necessary to find the commutators between the different components of angular momentum to understand the quantum version of angular momentum. Using the following facts,

$$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar \tag{5}$$

$$[x, y] = [y, z] = [z, x] = [p_x, p_y] = [p_y, p_z] = [p_z, p_x] = 0 \tag{6}$$

$$[x, p_y] = [x, p_z] = [y, p_x] = [y, p_z] = [z, p_x] = [z, p_y] = 0 \tag{7}$$

we obtain:

$$\begin{aligned}
[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\
&= [yp_z - zp_y, zp_x] - [yp_z - zp_y, xp_z] \\
&= [yp_z, zp_x] - [-zp_y, xp_z] \\
&= -i\hbar yp_x + i\hbar p_y x = i\hbar L_z
\end{aligned} \tag{8}$$

Similarly, we can easily obtain:

$$[L_y, L_z] = i\hbar L_x, \qquad [L_z, L_x] = i\hbar L_y \tag{9}$$

Notice that $[L_a, L_b] = i\hbar L_c$ if a, b, c is in a cyclic order. We can actually write this as

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k \tag{10}$$

where Einstein summation convention is assumed for repeated index k, even though it doesn't appear as an upper index and a lower index. Of course, we also assume i, j, k run from 1 to 3 and $L_1 = L_x, L_2 = L_y, L_3 = L_z$.

From these commutators, we can easily obtain the following relations:

$$[L_z, L_x + iL_y] = \hbar(L_x + iL_y) \tag{11}$$

$$[L_z, L_x - iL_y] = -\hbar(L_x - iL_y) \tag{12}$$

If we define as follows:

$$L_+ \equiv L_x + iL_y \tag{13}$$
$$L_- \equiv L_x - iL_y \tag{14}$$

we can rewrite the above formulas as follows:

$$[L_z, L_+] = \hbar L_+, \qquad [L_z, L_-] = -\hbar L_- \tag{15}$$

Moreover, let's define "L^2" as follows:

$$L^2 = L_x^2 + L_y^2 + L_z^2 \tag{16}$$

This corresponds to the square of the magnitude of angular momentum. Then, it is easy to check the followings:

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0 \tag{17}$$
$$[L^2, L_+] = [L^2, L_-] = 0 \tag{18}$$
$$L^2 = L_z^2 + L_z + L_- L_+ = L_z^2 - L_z + L_+ L_- \tag{19}$$

Now, let's consider a simultaneous eigenvector of L^2 and L_z. Such a vector always exists, as L^2 and L_z commute which implies that we can always find a basis in which the matrices L^2 and L_z are diagonal. Let's denote such a vector $|(l^2), m>$, where

$$L^2|(l^2), m> = l^2\hbar^2|(l^2), m> \tag{20}$$
$$L_z|(l^2), m> = m\hbar|(l^2), m> \tag{21}$$

Then, it is easy to see the followings:

$$\begin{aligned} L^2 L_+|(l^2), m> &= L_+ L^2|(l^2), m> \\ &= L_+ l^2\hbar^2|(l^2), m> \\ &= l^2\hbar^2 L_+|(l^2), m> \end{aligned} \tag{22}$$

Therefore, we conclude:

$$L^2(L_+|(l^2), m> = l^2\hbar^2(L_+|(l^2), m>) \tag{23}$$

In other words, $L_+|(l^2), m>$ is an eigenvector of L^2 with eigenvalues $l^2\hbar^2$. We also have:

$$\begin{aligned} L_z L_+|(l^2), m> &= (L_+ L_z + \hbar L_+)|(l^2), m> \\ &= L_+ m\hbar|(l^2), m> + \hbar L_+|(l^2), m> \\ &= (m+1)\hbar L_+|(l^2), m> \end{aligned} \tag{24}$$

Therefore, we conclude:

$$L_z(L_+|(l^2), m>) = (m+1)\hbar(L_+|(l^2), m>) \tag{25}$$

In other words, $L_+|(l^2), m>$ is an eigenvector of L_z with eigenvalues $(m+1)\hbar$.

Similarly,

$$L^2(L_-|(l^2), m>= l^2\hbar^2(L_-|(l^2), m>) \tag{26}$$

$$L_z(L_-|(l^2), m>) = (m-1)\hbar(L_-|(l^2), m>) \tag{27}$$

In other words, $L_-|(l^2), m>$ is an eigenvector of L^2 and L_z with eigenvalues of $l^2\hbar^2$ and $(m-1)\hbar$.

Now, given $|(l^2), m>$, we can apply multiple numbers of L_+s or L_-s to construct an eigenvector of L^2 and L_z with the same eigenvalue of L^2, but a different eigenvalue of L_z. For example,

$$L_z\left((L_+)^n|(l^2), m>\right) = (m+n)\hbar\left((L_+)^n|(l^2), m>\right) \tag{28}$$

$$L_z\left((L_-)^n|(l^2), m>\right) = (m-n)\hbar\left((L_-)^n|(l^2), m>\right) \tag{29}$$

In other words, $(L_+)^n|(l^2), m>$ is proportional to $|(l^2), m+n>$ and $(L_-)^n|(l^2), m>$ is proportional to $|(l^2), m-n>$.

At first glance, given an eigenvector $|(l^2), m>$, it may seem that one can construct other eigenvectors with L_z eigenvalues as high as or as low as we want, if we choose big enough n. But, this is troublesome, as the square of L_z should never be bigger than L^2 eigenvalues, as:

$$L^2 - L_z^2 = L_x^2 + L_y^2 \geq 0 \tag{30}$$

However, the formulas (28) and (29) are always satisfied. The only possibility is that $(L_+)^n|(l^2), m>$ is zero for big ns. Similarly for $(L_-)^n|(l^2), m>$. This implies that there exists k such that $(L_+)^k|(l^2), m>\neq 0$, but $(L_+)^{k+1}|(l^2), m>= 0$. Such a vector $(L_+)^k|(l^2), m>$ is called highest weight vector. Now, let's consider a highest weight vector $|(l^2), j>$. Then, we have:

$$L_+|(l^2), j>= 0 \tag{31}$$

which implies

$$0 = L_-L_+|(l^2), j>= (L^2 - L_z^2 - \hbar L_z)|(l^2), j> \tag{32}$$

$$= (l^2 - j^2 - j)\hbar^2|(l^2), j> \tag{33}$$

As $|(l^2), j>\neq 0$, we conclude

$$l^2 = j^2 + j = j(j+1) \tag{34}$$

In fact, it is customary to use the notation $|j,\ m>$ instead of $|(j(j+1)), m>$. From now on, we will use this notation. In other words,

$$L^2|j,\ m>= j(j+1)\hbar^2|j,\ m> \tag{35}$$

$$L_z|j,\ m>= m\hbar|j,\ m> \tag{36}$$

On the other hand, we can apply L_-s to $|j,\ j>$ to construct eigenvectors with lower L_z eigenvalues. Of course, this process should terminate at some point and the vector so obtained must be zero. Let's find how lowest the eigenvalue of L_z can be. Let's say $(L_-)^a|j,\ j>\neq 0$, while $(L_-)^{a+1}|j,\ j>=0$. Then, we have:

$$\begin{aligned} 0 &= L_+(L_-)^{a+1}|j,\ j>= L_+L_-(L_-)^a|j,\ j> && (37)\\ &= (L^2-L_z^2+\hbar L_z)(L_-)^a|j,\ j> && (38)\\ &= \big(j(j+1)-(k-a)^2+(k-a)\big)\,\hbar^2|j,\ j> && (39)\end{aligned}$$

Since $(L_-)^a|j,\ j>\neq 0$, we have:

$$j(j+1)=(j-a)^2-(j-a) \tag{40}$$

Therefore, we obtain:

$$a=2j,-1 \tag{41}$$

As a can't be negative we conclude $a=2j$. Now, notice that a should be an integer, as we cannot act "2.5" or "3.7" times of operator L_-. Therefore, we see that j must be non-negative half-integer as follows:

$$j=0,1/2,1,3/2,2\cdots \tag{42}$$

Now, what is the lowest possible value for L_z given j? From (29), we have:

$$L_z\left((L_-)^{2j}|j,\ j>\right)=-j\left((L_-)^{2j}|j,\ j>\right) \tag{43}$$

Here, we see that $(L_-)^{2j}|j,\ j>$ must be proportional to $|j,\ -j>$.

Finally, we see that the following vectors

$$|j,\ j>,\ |j,\ j-1>,\ |j,\ j-2>,\cdots,|j,\ -j+1>,\ |j,\ -j> \tag{44}$$

span the eigenvectors of L^2 with eigenvalue $j(j+1)$. It is also easy to see that this vector space is $(2j+1)$ dimensional.

Therefore, we conclude, in 3 space dimensions, both the square of magnitude of angular momentum and the value of the angular momentum along certain direction (for example $\hat{z}$ direction as considered in this article) are quantized.

Finally let me conclude this article with a couple of comments.

First, there are two types of angular momentum in quantum mechanics: spin and orbital angular momentum. Orbital angular momentum is due to the motion of particle while spin is not. Spin is an intrinsic angular momentum that has no classical counterparts. A particle has an angular momentum called spin even when it's not rotating. Also, different particles have different spins. For example, Higgs boson has spin 0, electrons and quarks have spin 1/2, photon has spin 1 and graviton has spin 2.

Second, one uses mathematics behind the angular momentum to derive the area spectrum in loop quantum gravity, even though the area spectrum has nothing to do with the angular momentum itself.

Problem 1. Check the following. (Hint[1])

$$< j,\ m|L_x|j,\ m >=< j,\ m|L_y|j,\ m >= 0 \tag{45}$$

Problem 2. Verify that the norm of the following vector is $\hbar\sqrt{j(j+1)-m(m+1)}$:

$$L_+|j,\ m > \tag{46}$$

(Hint[2])

Summary

- $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$. In other words, $[L_i, L_j] = i\epsilon_{ijk}L_k$.
- $L_+ = L_x + iL_y$ raises the eigenvalue of L_z by $\hbar$ and $L_- = L_x - iL_y$ lowers the eigenvalue of L_z by $\hbar$.
- L_+ and L_- don't change the eigenvalues of $L^2 = L_x^2 + L_y^2 + L_z^2$.
- L_z and L^2 commute. Therefore, we can diagonalize them in a common basis.
- $L^2|j,\ m\rangle = j(j+1)\hbar^2|j,\ m\rangle, \qquad L_z|j,\ m\rangle = m\hbar|j,\ m\rangle$.
- m can have values as $-j, -j+1, \cdots, j-1, j$.
- j can have values as $0, 1/2, 1, 3/2, 2, 5/2 \cdots$.

[1] Express L_x and L_y in terms of L_+ and L_-. Use also the fact that L_z is a Hermitian matrix which implies that two vectors with different eigenvalues for L_z are orthogonal to each other.

[2] The solution is in the next article.