Complex conjugate

In this article I will explain what a complex conjugate is. Notice that there is a certain arbitrariness in the definition of the imaginary number. Namely, there are two solutions to the equation $x^2 = -1$. The two solutions are i and $-i$. Suppose we replace i with $-i$. Then nothing would change as the "old" i would become the negative "new" i, while the negative "old" i would become the "new" i. Therefore, given an equation that includes the imaginary number i, nothing would change if we replace all the is with $-i$s. For example, here is an identity:

$$(1-2i)(1+4i) = 9+2i \tag{1}$$

Then, we can immediately see that

$$(1+2i)(1-4i) = 9-2i \tag{2}$$

Replacing i with $-i$ is called "complex conjugation." For example, "$2-3i$" is the complex conjugation of "$2+3i$." A complex conjugate is denoted by a bar or $*$. For example, the complex conjugate of z is denoted as either $\bar{z}$ or z^*.

An important property of the complex conjugate is that the complex conjugate of a real number is itself. For example, $3^* = 3$ or $-4.5^* = -4.5$. This is self-evident since the complex conjugate of $3 = 3+0i$ is $3-0i = 3$. The reasoning is similar for -4.5.

We can apply this property when examining the solution of equations with real coefficients. For example, if a, b, c, and d are real and if x is the solution to the following equation:

$$ax^3 + bx^2 + cx + d = 0 \tag{3}$$

then we can immediately see that $\bar{x}$ must be also the solution to this equation, since it must satisfy the following equation.

$$0 = \bar{a}\bar{x}^3 + \bar{b}\bar{x}^2 + \bar{c}\bar{x} + \bar{d} = a\bar{x}^3 + b\bar{x}^2 + c\bar{x} + d = 0 \tag{4}$$

Problem 1. Is the complex conjugate of the complex conjugate of a certain number same as itself? Explain.

Problem 2. Using the result of Problem 1, prove that $z + z^*$ is real for any complex number z.

Problem 3. Using the result of Problem 1, prove that $i(z - z^*)$ is real for any complex number z.

Problem 4. Prove that zz^* is real for any complex number z. Can it be negative? Explain.

Problem 5. Express $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ in terms of z and z^*.

Problem 6. If $z^* = -z$, what can you say about $\mathrm{Re}(z)$? Such z is called "purely imaginary." Justify this terminology.

Problem 7. One of the solution to $x^2 + bx + c = 0$ is $2 + i$. If b and c are real, what are they?

Problem 8. In the last article, we have seen

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = i \tag{5}$$

Using this result, can you find the solutions to the following equation? (Hint[1])

$$x^2 = -i \tag{6}$$

[1] Take the complex conjugate of (5).