Complex numbers

Let's try to solve the following equation:

$$x^2 = -9 \tag{1}$$

It is obvious that there is no solution to this equation. Certainly, no number squared is less than zero. Notice that the square of -3 is 9, not -9. Nevertheless, it is useful to introduce an imaginary number whose square is negative. Of course, such an imaginary number is not "real" in the usual sense. This is why it is called "imaginary." As a contrast, the ordinary number in usual sense is called "real" number.

Now, let's define an imaginary number i as follows:

$$i^2 = -1 \tag{2}$$

Then we can obtain the solutions to the equation (1) in terms of imaginary number as follows:

$$x = 3i, -3i \tag{3}$$

This is obvious since

$$(3i)^2 = 3^2 i^2 = 9 \times (-1) = -9 \tag{4}$$

The same holds for $-3i$.

If we use the concept of the imaginary number, solutions to quadratic equations always exist. For example:

$$x^2 + 2x + 5 = 0 \tag{5}$$

doesn't have any solution that is a real number. However, it has solutions if we don't restrict ourselves to the case of real solutions and use the concept of imaginary number as follows:

$$\begin{aligned}(x+1)^2 &= -4 \\ x + 1 &= 2i, \quad -2i \\ x &= -1 + 2i, \quad -1 - 2i \end{aligned} \tag{6}$$

Such a number as x, which is a sum of a real number and an imaginary number is called a "complex number." A complex number "z" can be expressed as follows:

$$z = x + iy \tag{7}$$

where x and y are real numbers. x is called the "real part" of z, and y the "imaginary part" of z.

You may wonder whether imaginary numbers exists, but they really do. Remember that I explained in the introduction to this article that imaginary numbers are widely used in quantum mechanics. Actually, apart from quantum mechanics, they are extensively used in science and engineering.

Notice that ancient people didn't recognize the existence of the negative number even though its existence is accepted now. The case with imaginary numbers is similar.

In the 16th century, a general method to solving cubic equations was obtained. A cubic equation is an equation that can be expressed as follows:

$$ax^3 + bx^2 + cx + d = 0 \tag{8}$$

where a is non-zero.

This equation can have one, two, or three real solutions. By real solutions, I mean solutions that are real numbers. However, in the case where there is only one real solution, you wouldn't be able to obtain this real solution without using the concept of imaginary numbers. The final result is a real number, but the intermediary step requires the use of the imaginary number to obtain the solution! Therefore, mathematicians began to realize that imaginary numbers really do exist.

Problem 1.

$$(1+i)(1-4i) =?, \qquad i^{10} =? \tag{9}$$

Problem 2.

$$\frac{1}{1+i} + \frac{1}{1-i} =?, \qquad \frac{1}{i^5} =? \tag{10}$$

Problem 3. Prove the following:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} \tag{11}$$

Problem 4. (Hint[1])

$$\left(\frac{1}{1+i}\right)^{10} =? \tag{12}$$

Problems 5. What is the real part of following? How about the imaginary part? (Hint[2])

$$\frac{1+5i}{1+i} \tag{13}$$

[1] $(1+i)^{10} = ((1+i)^2)^5$

[2] Use the result of Problem 3.