Complex numbers

Let's try to solve the following equation:

$$x^2 = -9 \tag{1}$$

It is obvious that there is no solution to this equation. Certainly, no number squared is less than zero. Notice that the square of -3 is 9, not -9. Nevertheless, it is useful to introduce an imaginary number whose square is negative. Of course, such an imaginary number is not "real" in the usual sense. This is why it is called "imaginary." As a contrast, the ordinary number in usual sense is called "real" number.

This should remind you of your first encounter with negative numbers. There is no solution to $x+4 = 3$ if we only consider non-negative solution for x. However, once we accept such x as existing, we can treat it as an actual number that satisfies basic properties of arithmetic such as $x + y = y + x$, $xy = yx$, and $x(y+z) = xy+xz$. Same goes for imaginary number. Once we accept imaginary number as existing, we can treat it as an actual number that satisfies basic properties of arithmetic.

Furthermore, the benefit of imaginary number is far-reaching. For example, it is very useful in the mathematical analysis of electric circuit, and is essential for quantum mechanics, as we will see in later articles.

As it is easy to forget the benefit of negative number, let us mention the following story as an aside before delving into explaining imaginary number in detail. In the early 16th century, Scipione del Ferro obtained a general solution to the cubic equation of the form $x^3 = px + q$ where p and q are positive numbers. At the time, mathematicians didn't have a firm concept of negative numbers. Therefore, del Ferro's solution couldn't be applied to solve the cubic equation of the form $x^3 + sx = t$, where s and t are positive numbers; these two cubic equations were regarded as two different types. Now, any smart high school student can solve the cubic equation of the form $x^3 + sx = t$ if she knows the general solution to $x^3 = px + q$; all she needs to do is plug in $p = -s$ and $q = t$ to the general solution of $x^3 = px+q$. However, this was impossible in the 16th century, because the substitution $p = -s$ required the concept of negative number.

Enough of an idle talk. Now, let's define an imaginary number i as follows:

$$i^2 = -1 \tag{2}$$

Then we can obtain the solutions to the equation (1) in terms of imaginary number as follows:

$$x = 3i, -3i \tag{3}$$

This is obvious since

$$(3i)^2 = 3^2 i^2 = 9 \times (-1) = -9 \tag{4}$$

The same holds for $-3i$.

If we use the concept of the imaginary number, solutions to quadratic equations always exist. For example:

$$x^2 + 2x + 5 = 0 \tag{5}$$

doesn't have any solution that is a real number. However, it has solutions if we don't restrict ourselves to the case of real solutions and use the concept of imaginary number as follows:

$$\begin{aligned} (x+1)^2 &= -4 \\ x + 1 &= 2i, \quad -2i \\ x &= -1 + 2i, \quad -1 - 2i \end{aligned} \tag{6}$$

Actually, we can check that they are the right solutions by substitution as follows. For $x = -1 + 2i$, we have

$$\begin{aligned} (-1+2i)^2 + (-1+2i) + 5 &= (-1)^2 + 2(-1)(2i) + (2i)^2 + 2(-1+2i) + 5 \\ &= 1 - 4i - 4 - 2 + 4i + 5 \\ &= 0 \end{aligned} \tag{7}$$

and similarly for $x = -1 - 2i$.

Such a number as x, which is a sum of a real number and an imaginary number is called a "complex number." A complex number "z" can be expressed as follows:

$$z = x + iy \tag{8}$$

where x and y are real numbers. x is called the "real part" of z, and y the "imaginary part" of z. We express this as $\mathrm{Re} z = x$, $\mathrm{Im} z = y$

You may wonder whether imaginary numbers exists, but they really do. Remember that I explained in the introduction to this article that imaginary numbers are widely used in quantum mechanics. Actually, apart from quantum mechanics, they are extensively used in science and engineering.

Notice that ancient people didn't recognize the existence of the negative number even though its existence is accepted now. The case with imaginary numbers is similar.

In the 16th century, a general method to solving cubic equations was obtained. A cubic equation is an equation that can be expressed as follows:

$$ax^3 + bx^2 + cx + d = 0 \tag{9}$$

where a is non-zero.

This equation can have one, two, or three real solutions. By real solutions, I mean solutions that are real numbers. However, in the case where there is only one real solution, it sometimes happens you wouldn't be able to obtain this real solution without using the concept of imaginary numbers. The final result is a real number, but the intermediary step requires the use of the imaginary number to obtain the solution! Therefore, mathematicians began to realize that imaginary numbers really do exist. In Problem 7, we will show you the actual example.

Problem 1.

$$(1+i)(1-4i) = ?, \qquad i^{10} = ?, \qquad \mathrm{Re}(3i) = ?, \qquad \mathrm{Im}(4) = ? \tag{10}$$

Problem 2.

$$\frac{1}{1+i} + \frac{1}{1-i} = ?, \qquad \frac{1}{i^5} = ? \tag{11}$$

Problem 3. Prove the following:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} \tag{12}$$

Problem 4. We have seen that the absence of the square root of a negative number motivated us to introduce imaginary numbers. Then, do we need to introduce another type of numbers to calculate the root of an imaginary number? Show that we don't, by checking

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = i \tag{13}$$

Thus, the square root of i can be expressed as a complex number. It is equal to $(1+i)/\sqrt{2}$. There are actually two complex numbers that satisfy $x^2 = i$. Can you find the other one? (Hint[1])

Problem 5. (Hint[2])

$$\left(\frac{1}{1+i}\right)^{10} = ? \tag{14}$$

Problem 6. What is the real part of following? How about the imaginary part? (Hint[3])

$$\frac{1+5i}{1+i} \tag{15}$$

Problem 7. In 1572, Bombelli published a solution to $x^3 = 15x + 4$. For one of the real solutions, he obtained

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} \tag{16}$$

[1] If x satisfies $x^2 = i$, it also satisfies $(-x)^2 = i$.

[2] $(1+i)^{10} = ((1+i)^2)^5$

[3] Use the result of Problem 3.

Show the following: (Hint[4])

$$\sqrt[3]{2+\sqrt{-121}} = 2+i, \qquad \sqrt[3]{2-\sqrt{-121}} = 2-i \tag{17}$$

Thus,

$$x = 2+i+2-i = 4 \tag{18}$$

[4]Show $(2+i)^3 = 2+\sqrt{-121}$ and $(2-i)^3 = 2-\sqrt{-121}$.