Complex numbers and the trigonometric functions

Is there a good way to visualize complex numbers? The answer is yes, and it's very useful. First, notice that we need two real numbers to express one complex number. For example, a complex number $3 + 5i$ requires two real numbers 3 and 5. Therefore, we can represent a complex number on two-dimensional Cartesian coordinate. A conventional choice is to represent the real part of a complex number on x-axis while the imaginary part on y-axis as in Fig.1.

However, instead of Cartesian coordinate, we may use polar coordinate and it turns out to be useful. For example, instead of "a", the real part of the number, and "b" the imaginary part as in Fig.1, we can express it in terms of "r" the "length" of the complex number and "θ" the angle it makes with x-axis as in Fig.2.

Of course, we have the following identity.

$$r = \sqrt{a^2 + b^2} \tag{1}$$

$$\tan\theta = \frac{b}{a} \tag{2}$$

$$a = r\cos\theta \tag{3}$$

$$b = r\sin\theta \tag{4}$$

As a side remark, the "length" of a complex number z is called the "magnitude" or the "absolute value" of z, and denoted by $|z|$. It is an easy exercise to check the following relation.

$$|z|^2 = z\bar{z} \tag{5}$$

You can prove this by substituting $z = a + bi$ and express everything in terms of a and b.

"θ" of a complex number z is called the "argument" or the "phase" of z. Using these two variables "r" and "θ", one can express a complex number "z" as "$r\text{cis }\theta$." Of course, we have the following relation.

$$r\text{cis }\theta = r(\cos\theta + i\sin\theta) \tag{6}$$

Check also yourself that $|r\text{cis }\theta| = |r|$

Now I will explain the addition and the multiplication of two numbers in this pictorial representation of complex numbers. First of all, the addition is straightforward. The addition of "$z_1 = a + bi$" with "$z_2 = c + di$" is simply "$z_1 + z_2 = (a + c) + (b + d)i$" and depicted in Fig.3.

The multiplication is more interesting. However, before presenting the pictorial representation of the multiplication of two numbers, we have to find out how multiplication works in the polar coordinate representation as it is more relevant in this case. Let two complex numbers z_1, z_2 be defined as follows.

$$z_1 = r_1 \text{cis } \theta_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad (7)$$

$$z_2 = r_2 \text{cis } \theta_2 = r_2(\cos\theta_2 + i\sin\theta_2) \quad (8)$$

Then, a straightforward calculation shows the following.

$$z_1 * z_2 = r_1 * r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \quad (9)$$

However, surprisingly, the above formula can be simplified as follows:

$$z_1 * z_2 = r_1 * r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \quad (10)$$

where we have used the addition rule of trigonometric functions. Pictorially, this can be represented as in Fig.4.

Therefore, multiplying by z_2 is equivalent to multiplying the absolute value by the absolute value of z_2, and rotating the phase by the phase of z_2. In other words, we found the following coincidence:

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \quad (11)$$

For example, if you multiply a complex number by i it is equivalent to rotating the complex number anticlockwise by 90 degree, since $\text{cis}90^\circ = i$. Similarly, if you multiply a complex number by -1, it is equivalent to rotating the complex number anticlockwise by 180 degrees since $\text{cis}180^\circ = -1$. This makes sense since multiplying by i twice is equivalent to rotating anticlockwise by 90 degrees twice which is equivalent to rotating 180 degrees which is equivalent to multiplying by -1.

Let me explain why there was such a coincidence. The mathematician Euler proved the following:

$$\cos\theta + i\sin\theta = e^{i\theta} \quad (12)$$

where e is a number that is very important in mathematics and approximately given by $2.718\cdots$. It may seem a little bit strange that the exponent could be an imaginary number, but there is a nice way to define it. The coincidence we found can be easily derived since

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \quad (13)$$

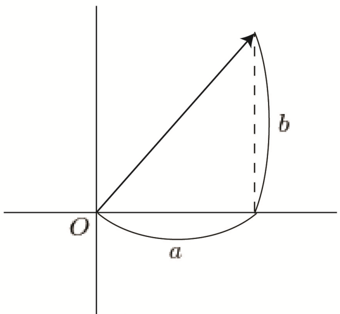

Fig. 1

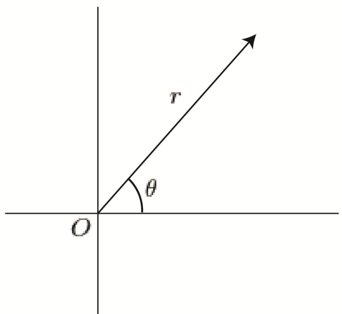

Fig. 2

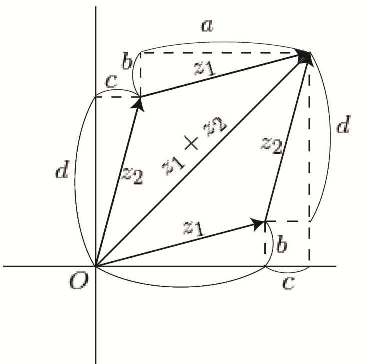

Fig. 3

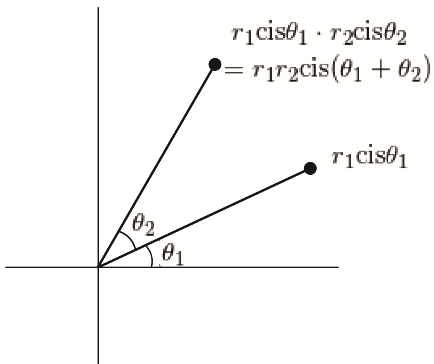

Fig. 4

This is obvious as

$$e^a e^b = e^{a+b} \tag{14}$$

Proof of Euler's formula requires an advanced knowledge of calculus. Interested readers may find it in Wikipedia or another article of mine.

Problem 1. Check $|z|^2 = z\bar{z}$ and $|r\text{cis }\theta| = |r|$

Problem 2. Check

$$\frac{1+\sqrt{3}i}{2} = \text{cis } 60^\circ \tag{15}$$

Using this, obtain the value for following

$$\left(\frac{1+\sqrt{3}i}{2}\right)^{10} \tag{16}$$

Problem 3. Find all solutions to $z^8 = 1$ where z is a complex number. (Hint[1])

[1]Let $z = r\text{cis}\theta$ and use $1 = \text{cis } 0^\circ = \text{cis } 360^\circ = \text{cis } 720^\circ = \cdots = \text{cis } (7 \times 360^\circ)$