# Composition

A composition is an integer partition in which order is taken into account. For example, there are eight compositions of 4: $4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2$ and $1+1+1+1$. $c(n)$ denotes the number of compositions of $n$, and $c_m(n)$ is the number of compositions into exactly $m$ parts. For example: $c(4) = 8$, $c_3(4) = 3$.

It is easy to understand that $c_m(n)$ is given by the coefficient of $x^n$ in the expansion of

$$(x + x^2 + x^3 + \cdots)^m \tag{1}$$

Let's explicitly see this, in the case of $c_3(4) = 3$

$$\begin{aligned} & (x + x^2 + x^3 + \cdots)^3 && (2) \\ = \; & (x + x^2 + x^3 + \cdots)(x + x^2 + x^3 + \cdots)(x + x^2 + x^3 + \cdots) \\ = \; & x \cdot x \cdot x + x^2 \cdot x \cdot x + x \cdot x^2 \cdot x + x \cdot x \cdot x^2 + \cdots && (3) \\ = \; & 1 \cdot x^3 + 3x^4 + \cdots && (4) \end{aligned}$$

So, we see that $c_3(3)$ is 1 and $c_3(4)$ is indeed 3. In other words:

$$(x + x^2 + x^3 + \cdots)^3 = c_3(3)x^3 + c_3(4)x^4 + \cdots \tag{5}$$

More generally, we can write

$$\sum_{m=1}^{\infty} (x + x^2 + x^3 + \cdots)^m = \sum_{n=1}^{\infty} c_m(n)x^n \tag{6}$$

Furthermore, it is also easy to see that

$$c(n) = \sum_{m=1}^{\infty} c_m(n) \tag{7}$$

since the number of compositions of $n$ is the sum of the number of compositions of $n$ into exactly 1 part, 2 parts, 3 parts, 4 parts and so on. So, we have:

$$\sum_{m=1}^{\infty} (x + x^2 + x^3 + \cdots)^m = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m(n)x^n = \sum_{n=1}^{\infty} c(n)x^n \tag{8}$$

Now, we can calculate $c(n)$. Recalling our earlier article on the sum of geometric series, the above formula is equal to:

$$\frac{(x + x^2 + x^3 + \cdots)}{1 - (x + x^2 + x^3 + \cdots)} = \sum_{n=1}^{\infty} c(n)x^n \tag{9}$$

$$= \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} = \frac{x}{1 - 2x}$$

$$= x(1 + 2x + (2x)^2 + (2x)^3 + \cdots) = \sum_{n=1}^{\infty} 2^{n-1}x^n \tag{10}$$

Therefore, we conclude $c(n) = 2^{n-1}$. This agrees with our earlier result $c(4) = 8$.

## Summary

- A composition is an integer partition in which order is taken into account.
- $c(n)$ denotes the number of compositions of $n$, and $c_m(n)$ is the number of compositions into exactly $m$ parts.
- $c_m(n)$ is given by the coefficient of $x^n$ in the expansion of $(x + x^2 + x^3 + \cdots)^m$.