Composition of functions and inverse functions

Consider you have two functions f and g, and you enter a number x to g. Then, enter its output $g(x)$ to f. Then, as a final result, you will get $f(g(x))$. However, you can regard this as a single function; if you enter a number x, you get $f(g(x))$. If we call this function $h(x)$, we have $h(x) = f(g(x))$. This is the concept of "composition of functions." Mathematicians often express $h(x) = f(g(x))$ as $h = f \circ g$. Notice also that you first apply g then f, but you write this in the inverse order, namely $h = f \circ g$. This may be done to match the order $f(g(x)) = f \circ g(x)$

(**Problem 1.** Let $f(x) = x^2 - 6x$, and $g(x) = x + 3$. Find $f \circ g$.)

Now, suppose you have a function h and you enter a number x to h. Then, a certain number $h(x)$ will pop out. Suppose you want to reverse the function h; if $h(x)$ is entered, you get x. We call such a function an "inverse function." The inverse function of h is denoted as h^{-1}. For example, we have $h(h^{-1}(x)) = x$ by definition. However, we have to be careful when dealing with the inverse function. For example, consider $h(x) = x^2$. Then, we know that there are two inverse functions $h^{-1}(x) = \sqrt{x}$ and $h^{-1}(x) = -\sqrt{x}$. When defining the inverse function, we have to manually choose which one we want.

Problem 2. Let $f(x) = \sqrt{x} - 3$. Find $f^{-1}(x)$.

Problem 3. Let $f(x) = x - 3$ and $g(x) = 2x + 4$. Find $f \circ g$ and $g \circ f$.

Problem 4. Let $f \circ g(x) = x^2 - 1$, and $g(x) = x - 3$. Find $f(x)$. (Hint[1])

Problem 5. Let $f \circ g(x) = x^2 - 4x + 8$, and $f(x) = x^2 + 4$. Find all possible $g(x)$. (Hint[2])

Summary

- If $h(x) = f(g(x))$, $h = f \circ g$. This is the composition of two functions.
- The inverse function of f is denoted by f^{-1}. It is defined by $f(f^{-1}(x)) = x$.

[1] Use $f \circ g \circ g^{-1} = f$.

[2] Notice $f(g(x)) = (x-2)^2 + 4 = (2-x)^2 + 4$.