Continuity equation

Instead of air volume, consider now the flux of air mass, assuming that the air density ρ doesn't need to be constant. Then the flux through a surface $\Delta\vec{A}$ is given by

$$\Phi = \rho\vec{v} \cdot \Delta\vec{A} \tag{1}$$

since this would be the air mass that passes through this surface during unit time.

Now, notice that the total flux through a closed surface doesn't necessarily have to be zero, since ρ is not constant. In case of air volume, the total flux must be zero, since the closed surface encompasses only a constant volume, which must be equal to the total air volume. In the air mass case, the change of M, the total air mass inside the closed surface would be given by the flux as follows.

$$\frac{dM}{dt} = -\oint (\rho\vec{v}) \cdot d\vec{A} \tag{2}$$

We have a negative sign here, since the total mass of the air decreases, if the air comes out of the surface, in which case the flux is positive.

Using the fact that the mass is given by the integration of density over volume, we can re-express the above equation as follows:

$$\int \frac{\partial\rho}{\partial t} dV = -\oint (\rho\vec{v}) \cdot d\vec{A} \tag{3}$$

This is called “continuity equation.” It signifies the fact that air is neither destroyed nor generated, and every increment of air inside a closed surface is due to the movement of air passing through the surface. In this article, we considered the air but it could be any other things, such as electric charge or fluid.

Summary

- From the fact that a certain substance can be neither destroyed nor generated, but can only flow from one place to the other, one can derive the continuity equation which relates the time derivative of its mass density with its velocity. The concept of flux is crucial in its derivation.