Coupled harmonic oscillator system

You already know how to obtain the solution to the equation of motion when there is one spring and one object. However, what you do not know yet is how to obtain the equation of motion when there are multiple springs and multiple objects. For example, see Fig.1. We have three springs with each spring constant k and two objects with each mass m. What will be the equation of motion? Let's choose the coordinates in such a way that the positions $x_1 = x_2 = 0$ are at equilibrium. Denoting $\frac{d^2x}{dt^2}$ by $\ddot{x}$, we have:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \tag{1}$$

$$m\ddot{x}_2 = -kx_2 + k(x_1 - x_2) \tag{2}$$

We have to solve these equations, but we do not know yet how. But, we can try. From the symmetry of configurations, (i.e. all the springs have the same spring constant and all the objects have the same mass) we try summing (1) and (2). Then, we get:

$$m\frac{d^2(x_1 + x_2)}{dt^2} = -k(x_1 + x_2) \tag{3}$$

Therefore, we have

$$x_1 + x_2 = 2A\sin(\omega_A t + \theta_A) \tag{4}$$

where $\omega_A = \sqrt{k/m}$, and the coefficient 2 in front of A is for future convenience.

Now, let's try subtracting (2) from (1). We have:

$$m\frac{d^2(x_1 - x_2)}{dt^2} = -3k(x_1 - x_2) \tag{5}$$

Therefore, we have

$$x_1 - x_2 = 2B\sin(\omega_B t + \theta_B) \tag{6}$$

where $\omega_B = \sqrt{3k/m}$, and the coefficient 2 in front of B is again for future convenience.

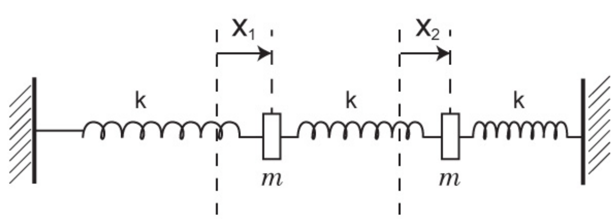

Figure 1: 2 objects and 3 springs

Now, from (4) and (6), we can obtain:

$$x_1 = A\sin(\omega_A t + \theta_A) + B\sin(\omega_B t + \theta_B) \tag{7}$$

$$x_2 = A\sin(\omega_A t + \theta_A) - B\sin(\omega_B t + \theta_B) \tag{8}$$

Now, let's interpret these equations. We have two oscillating modes present in the above equations. The first one is given in terms of ω_A and the second one ω_B. From the coefficients (i.e. A and A), the first one corresponds to $x_1 = x_2$. Similarly, from the coefficients (i.e. B and $-B$) the second one corresponds to $x_2 = -x_1$.

Let's closely look at them again. For the first case, the position difference between the two objects is constant; the middle spring is neither compressed nor stretched, and can be regarded as not existing in this mode. Plugging $x_2 = x_1$ to (1) and $x_1 = x_2$ to (2), we get:

$$m\ddot{x}_1 = -kx_1 \tag{9}$$

$$m\ddot{x}_2 = -kx_2 \tag{10}$$

Solving the above equations, with the condition $x_1 = x_2$, we have

$$x_1 = A\sin(\omega_A t + \theta_A) \tag{11}$$

$$x_2 = A\sin(\omega_A t + \theta_A) \tag{12}$$

where $\omega_A = \sqrt{k/m}$. Notice that they are oscillating with the same frequency. Otherwise, $x_1 = x_2$ cannot be satisfied.

For the second case, plugging $x_2 = -x_1$ to (1) and plugging $x_1 = -x_2$ to (2) yields:

$$m\ddot{x}_1 = -3kx_1 \tag{13}$$

$$m\ddot{x}_2 = -3kx_2 \tag{14}$$

Solving the above equations, with the condition $x_2 = -x_1$, we have:

$$x_1 = B\sin(\omega_B t + \theta_B) \tag{15}$$

$$x_2 = -B\sin(\omega_B t + \theta_B) \tag{16}$$

where $\omega_B = \sqrt{3k/m}$. Notice that they are oscillating with the same frequency. Otherwise, $x_2 = -x_1$ cannot be satisfied.

Now, we can once again check that the general solutions (7) and (8) are the sum of (11) and (15), and the sum of (12) and (16).

Now, think about how we could solve this problem without making guesses such as $x_1 + x_2$ and $x_1 - x_2$. What we have just done is finding modes in which x_1 and x_2 oscillate together with the same frequency. Then, the general solution is given by the sum of these modes. To find the frequency, let's write:

$$x_1 = C_1\sin(\omega t + \theta) \tag{17}$$

$$x_2 = C_2\sin(\omega t + \theta) \tag{18}$$

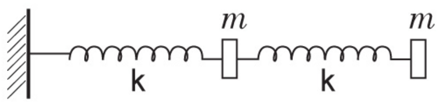

Figure 2: 2 objects and 2 springs

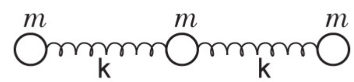

Figure 3: 3 objects and 2 springs

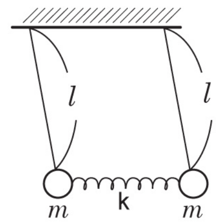

Figure 4: 2 objects, 2 pendulums and 2 springs

Then, we have:

$$\ddot{x_1} = -\omega^2 x_1 \tag{19}$$

$$\ddot{x_2} = -\omega^2 x_2 \tag{20}$$

We can plug them into (1) and (2). Then, we have:

$$(m\omega^2 - 2k)x_1 + kx_2 = 0 \tag{21}$$

$$kx_1 + (m\omega^2 - 2k)x_2 = 0 \tag{22}$$

Notice that this is a system of linear equations. If you choose an ω which is not quite special, the solution to the above equation would be $x_1 = x_2 = 0$ as the ratio $(m\omega^2 - 2k)$ to k would not be equal to the ratio k to $(m\omega^2 - 2k)$ and this solution would be meaningless as the objects not oscillating at all is really meaningless. To have a non-zero solution, we must have

$$\begin{vmatrix} (m\omega^2 - 2k) & k \\ k & (m\omega^2 - 2k) \end{vmatrix} = 0 \tag{23}$$

In other words, we are solving:

$$\frac{x_2}{x_1} = -\frac{m\omega^2 - 2k}{k} = -\frac{k}{m\omega^2 - 2k} \tag{24}$$

This is a quadratic equation and if you solve this, you will get $m\omega^2 = k,\ 3k$. Plugging $m\omega^2 = k$ to (21) and (22), you get $x_1 = x_2$. Plugging $m\omega^2 = 3k$ to (21) and (22), we get $x_1 = -x_2$.

I learned this method at Korean Physics Olympiad camp when I was in middle school. Then, I didn't know why there are non-zero solutions when the determinant is zero as the teacher there didn't explain it, but I learned it just as a trick, and I accepted it.

Problem 1. Show that (23) is actually an eigenvalue problem.

Problem 2. See Fig. 2. Write down the equation of motion and obtain a general solution to it.

Problem 3. See Fig. 3. Write down the equation of motion and obtain a general solution to it.

Problem 4. See Fig. 4. For small oscillations, write down the equation of motion and obtain a general solution to it.

Summary

- When more than one object is connected to one another through springs, we can find their general equations of motion by solving an eigenvalue problem in linear algebra.
- In particular, we first need to find the eigenmodes, the certain linear combinations of the positions of objects, which behave like a simple harmonic oscillator.
- After finding the eigenmodes, we need to re-express the positions of objects in terms of the linear combination of the eigenmodes. Then, we are done.