De Rham cohomology, homology and Künneth formula

In an earlier article, I introduced differential forms. In this article, I will explain how it is related to topology. Readers who aren't satisfied with this article or want to learn more about this topic can consult Chapter 6 of "Topology and Geometry for Physicists" by Charles Nash and Siddhartha Sen.

Now, let me introduce closed forms and exact forms. A closed form w is a form such that

$$dw = 0 \tag{1}$$

An exact form w is a form such that

$$w = df \tag{2}$$

for some f.

As $d^2 = 0$, we have

$$d^2 f = d(df) = dw = 0 \tag{3}$$

This implies that an exact form is always a closed form, but not vice versa. De Rham cohomology tells us how many more closed forms there are than exact forms. This depends on the topology of the background where these differential forms are defined.

To understand this, let's consider a torus. See Fig.1.

As a torus is two-dimensional, we can assign two coordinates to specify a point on this torus. (When we say a torus in mathematics, we mean the surface of the torus, not the interior. Therefore, it's two-dimensional rather than three-dimensional) Let's denote these coordinates as θ_1 and θ_2, and also say that each of them range from 0 to 2π. Of course the point $\theta_1 = 0$ is the same point as $\theta_1 = 2\pi$, and similarly for θ_2. Given this, what would be the one-forms on this torus? Naturally, we can write a one-form w as following:

$$w = f(\theta_1, \theta_2)d\theta_1 + g(\theta_1, \theta_2)d\theta_2 \tag{4}$$

In this case, there are two bases for one-form: $d\theta_1$ and $d\theta_2$.

These one-forms are closed. Now let's see whether they are exact. If they are exact, we can write:

$$d\theta_1 = d(\alpha_1) \tag{5}$$

$$d\theta_2 = d(\alpha_2) \tag{6}$$

for some α_1 and α_2. Notice that $d\theta_1$ doesn't necessarily mean $d(\theta_1)$, even though this notation may be confusing. This confusion should be cleared out as you further read this article.

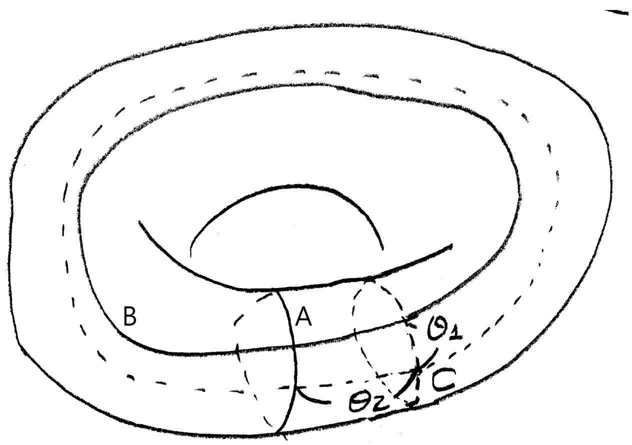

Figure 1: Coordinates θ_1 and θ_2

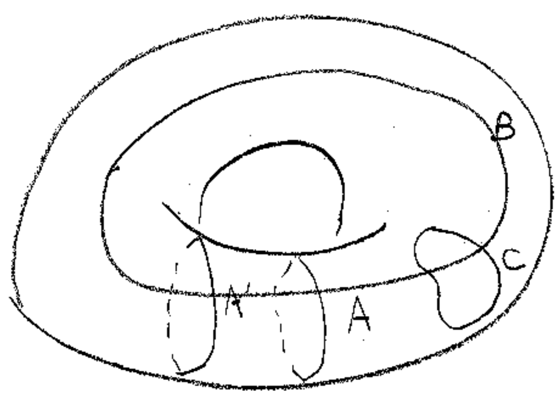

Figure 2: A, A', B, C

Now, let's calculate the integration of each of the one-forms on the following two circles A, and B denoted on the Fig.2. Then using Stoke's theorem, we can write:

$$\int_0^{2\pi} d\theta_1 = \int_A d\theta_1 = \int_A d(\alpha_1) = \int_{dA} \alpha_1 = 0 \tag{7}$$

where we have used the fact that a circle has no boundary.

Similarly, we can write

$$\int_0^{2\pi} d\theta_2 = \int_B d\theta_2 = \int_B d(\alpha_2) = \int_{dB} \alpha_2 = 0 \tag{8}$$

However, the above integrals should not be zero as

$$\int_0^{2\pi} d\theta_1 = \int_0^{2\pi} d\theta_2 = 2\pi \tag{9}$$

Therefore, $d\theta_1$ and $d\theta_2$ are examples of one-form that is closed but not exact. Notice that this happened because the coordinates θ_1 and θ_2 are periodic. This is unavoidable if we try to define coordinates on torus. Also notice that this unavoidability depends on the topology of the background in which the differential forms are defined.

We call $d\theta_1$ and $d\theta_2$ the de Rham cohomology elements of H^1_{dR}(Torus), where 1 denotes one in the one-form, and dR denotes de Rham. As there are two linearly independent elements in H^1_{dR}(Torus), we say $H^1_{dR}(\text{Torus}) = \mathbb{R}^2$ where 2 here denotes the number of linearly independent elements.

Then, what would be H^0_{dR}(Torus)? A constant function c satisfies $dc = 0$ which means that it's closed but it's not exact since there is no object b that would satisfy $db = c$ since b must be "-1 form" and there is no such thing as "-1 form." Therefore, a constant function is the basis of the de Rham cohomology of 0-form, and is the only basis. So,

$$H^0_{dR}(\text{Torus}) = \mathbb{R}^1 \tag{10}$$

Then, what would be H^2_{dR}(Torus)? Naturally, consider the fact that any two-forms must be expressed in terms of the wedge product of two one-forms. Therefore, we can denote a two form, β as:

$$\beta = k(\theta_1, \theta_2) d\theta_1 \wedge d\theta_2 \tag{11}$$

So the basis of the two-form in torus is $d\theta_1 \wedge d\theta_2$. It is easy to see that it is closed.

$$d(d\theta_1 \wedge d\theta_2) = dd\theta_1 \wedge d\theta_2 - d\theta_1 \wedge dd\theta_2 = 0 \wedge d\theta_2 - d\theta_1 \wedge 0 = 0 \tag{12}$$

However, it is not exact, since neither $d\theta_1$ nor $d\theta_2$ is exact. (If they were, we would be able to express $d\theta_1 \wedge d\theta_2$ as $d(\theta_1 \wedge d\theta_2)$ or $d(-d\theta_1 \wedge d\theta_2)$. Therefore, H^2_{dR}(Torus) is $\mathbb{R}^1$.

Now, what would be the de Rham cohomology elements of $\mathbb{R}^n$? ($\mathbb{R}^n$ is the usual n-dimensional Euclidean space.)

$H^0_{dR}(\mathbb{R}^n) = \mathbb{R}^1$ from the same reason as $H^0_{dR}(\text{Torus}) = \mathbb{R}^1$. However, $H^k_{dR}(\mathbb{R}^n) = \mathbb{R}^0$ for k other than 0 since the natural variables specifying the position of R^n can be always assigned non-periodic.

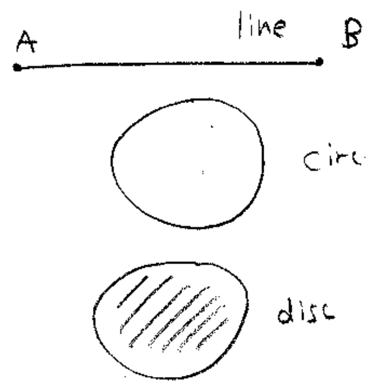

Figure 3: boundaries of line, circle and disc

There is an equivalent way of denoting the information that de Rham cohomology carries without using the language of differential forms. It is homology. Homology also uses the language of being closed and being exact. The difference is that it is with respect to the boundary operator for homology unlike de Rham cohomology which is with respect to exterior derivative. The natural objects are also different. In case of homology, it's a n-dimensional object while it's differential forms in case of de Rham cohomology.

So what is boundary operator? This is the same boundary operator that appears in Green's theorem and Stoke's theorem. For examples, see Fig.3. The boundary of a simple open line is two points, A and B. There is no boundary for a circle. Also, the boundary of the boundary of an object is always vanishing. For example, the boundary of a disc is a circle whose boundary vanishes.

Therefore, the boundary of n-dimensional object which is a boundary of something is always vanishing, but not vice versa. In other words, we can say as before, an exact object is always closed but not vice versa.

As in the case of de Rham cohomology, homology counts the elements which are closed but not exact. See Fig.2. In case of torus, the circle A, the circle A' and the circle B are such examples, as they aren't boundaries of any 2-dimensional objects, yet they have no-boundaries. (It may seem that circle A is a boundary of a 2-dimensional object, but remember? In mathematics, torus means the 2-dimensional surface of the torus, not the 3-dimensional interior.) Also, notice that loop C is an example which is exact and closed, so it doesn't count as a homology element. Moreover, in homology we count circle A' as the same element as the circle A since circle A can be continuously moved to circle A without breaking or re-joining it. (This is not a exact statement, but rather a rough statement. Precisely speaking, the difference between A and A' (i.e. $A - A'$) is a boundary of two-dimensional surface (the one enclosed by A and A'), so they count as same.) For example, it is easy to see that circle A and loop C are different, as you need to cut circle A and reconnect it to make it loop C. You have to "unwind" A that wraps the torus.

Therefore, the conclusion is that $H_1(\text{Torus})$ is $\mathbb{Z}^2$, as there are exactly two one-dimensional objects which are closed, but non-exact. Namely, A and B. Here $\mathbb{Z}$ means integer, unlike the case of the de Rham cohomology which usually uses $\mathbb{R}$, the real number. Of course, in most cases, this is rather just a notational difference without deep mathematical meanings.

Notice that the number above 2 is the same as the number 2 in $H^1_{dR}(\text{Torus}) = \mathbb{R}^2$. I hope that the reader intuitively feels that $H_k(M)$ is $\mathbb{Z}^j$ if $H^k_{dR}(M)$ is $\mathbb{R}^j$; there is one to one correspondence between closed but non-exact k-form and closed but non-exact k-dimensional object. In our torus case, $d\theta_1$ corresponds to A and $d\theta_2$ corresponds to B. In other words, homology and de Rham cohomology contain the same information. Of course, it may be hard to prove this, even though this is intuitively clear. De Rham proved this in 1931.

Now, let's calculate de Rham cohomology (or equivalently, homology) of one more complicated example. As a first step to this end, notice that a torus is the direct product of two circles. In other words, a point in torus is specified by its location in the first circle A and its location in the second circle B. Recall that they were θ_1 and θ_2. We express this fact as

$$\text{torus} = S^1 \times S^1 \tag{13}$$

where S^1 denotes the circle. Similarly, we can define 4-dimensional torus T^4 by $S^1 \times S^1 \times S^1 \times S^1$.

Now, let's calculate the de Rham cohomology of this object. First of all, $H^0_{dR}(T^4) = \mathbb{R}^1$ from the same reason as before; its element is a constant function. Second, $H^1_{dR}(T^4) = \mathbb{R}^4$. The four elements are $d\theta_1, d\theta_2, d\theta_3, d\theta_4$. Third, $H^2_{dR}(T^4) = \mathbb{R}^6$. The six elements are $d\theta_1 \wedge d\theta_2, d\theta_1 \wedge d\theta_3, d\theta_1 \wedge d\theta_4, d\theta_2 \wedge d\theta_3, d\theta_2^d\theta_4$, and $d\theta_3 \wedge d\theta_4$. Fourth, $H^3_{dR}(T^4) = \mathbb{R}^4$. The four elements are $d\theta_1 \wedge d\theta_2 \wedge d\theta_3, d\theta_2 \wedge d\theta_3 \wedge d\theta_4, d\theta_1 \wedge d\theta_3 \wedge d\theta_4, d\theta_1 \wedge d\theta_2 \wedge d\theta_3$. Finally, $H^4_{dR}(T^4) = \mathbb{R}^1$. The only element is $d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4$.

Finally, let me explain what Künneth formula is. If an n-dimensional object M of which the de Rham cohomology we want to calculate is direct product of two smaller dimensional objects, say k-dimensional object X and $(n-k)$ dimensional object Y, the de Rham cohomology of M is given by the following formula:

$$H^a_{dR}(M) = \sum_{a=b+c} H^b_{dR}(X) \times H^c_{dR}(Y) \tag{14}$$

Let's see why this should be the case. Let's say $a = 3$. Then, we can easily see that a 3-form on M can be expressed as sums of a wedge product of 0-form in X with a 3-form in Y and a wedge product of 1-form in X with 2-form in Y and a wedge product of 2-form in X with 1-form in Y and a wedge product of 3-form in X and 0-form in Y. This is not a rigorous proof, but a rough argument. Künneth formula was rigorously proved by Künneth in his doctoral thesis in 1922.

Summary

- A closed form w is a form such that $dw = 0$.
- An exact form w is a form such that $w = df$ for some f.
- $d^2 = 0$ implies, an exact form is always a closed form.
- De Rham cohomology tells us how many more closed forms there are than exact forms.
- In the case of homology, the natural object is an n-dimensional manifold while the natural objects are differential forms in case of de Rham cohomology.
- ∂ is called the "boundary operator."
- A cycle f satisfies $\partial f = 0$. A boundary f satisfies $f = \partial g$ for some manifold g.
- Homology tells us how many more cycles there are than boundaries.