

The determinant

In the last article, we gave you a formula for the determinant of 2×2 matrices. In this article, we will first give you the properties of determinant and show you that it satisfies the algebraic property of determinant explained in the last article. Then, give you an explicit formula for the determinant.

A determinant gives you a number for $n \times n$ matrix. However, you can think of $n \times n$ matrix as n n -dimensional vectors. For example, consider the following 3×3 matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 3 & -3 & 2 \end{bmatrix} \quad (1)$$

This can be understood as collections of the following three vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad (2)$$

Given this, we can understand that determinant is a function a number is spit out if n n -dimensional vectors are entered as follows.

$$\det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = \text{number} \quad (3)$$

Now, let's enumerate the properties of the determinant. The first property of determinant is "multilinearity." If you remember our earlier article "Matrices and Linear Algebra," you will remember linearity. Determinant respects this property for each vector entered. For example,

$$\begin{aligned} \det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \alpha \vec{v}_i, \vec{v}_{i+1}, \dots, \vec{v}_n) &= \alpha \det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_i, \vec{v}_{i+1}, \dots, \vec{v}_n) \\ &= \det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{u}_i + \vec{w}_i, \vec{v}_{i+1}, \dots, \vec{v}_n) \\ &= \det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{u}_i, \vec{v}_{i+1}, \dots, \vec{v}_n) + \det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{w}_i, \vec{v}_{i+1}, \dots, \vec{v}_n) \end{aligned}$$

The second property is antisymmetry. If you exchange two vectors among n -vectors entered, the determinants get an extra negative sign. For example,

$$\det(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n) = -\det(\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n) \quad (4)$$

A corollary to this property is that if the same vector is repeated in the determinant, it is zero. For example, if we plug $\vec{u} = \vec{v}_i = \vec{v}_j$ to the above equation, we have:

$$\det(\vec{v}_1, \dots, \vec{u}, \dots, \vec{u}, \dots, v_n) = -\det(\vec{v}_1, \dots, \vec{u}, \dots, \vec{u}, \dots, \vec{v}_n) \quad (5)$$

which implies:

$$\det(\vec{v}_1, \dots, \vec{u}, \dots, \vec{u}, \dots, v_n) = 0 \quad (6)$$

The third property is normalization. Determinant of identity matrix is 1. In other words,

$$\det I = 1 \quad (7)$$

These three properties uniquely determine determinant. You will be convinced of this statement when we will revisit determinant in our late article “The determinant and its geometric interpretation.”

Now, let me prove the claim that I made in our earlier article. There, we had the following equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

This means:

$$x \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + y \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + z \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

If we let

$$\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \quad (10)$$

we can re-express (9) as follows:

$$x\vec{a}_1 + y\vec{a}_2 + z\vec{a}_3 = 0 \quad (11)$$

Now, what we want to show is that the determinant of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is zero if there are non-zero solutions to the above equation. (i.e., if these three vectors are linearly dependent.) In other words, a solution $x, y,$ and z are not simultaneously zero. Let’s consider such a case and let’s assume that x is not zero. (If x is zero, we can choose y . If y is also zero, we can choose z , which must not be zero if both x and y are zero, since all three can’t be simultaneously zero. Then, the argument will go the same.) Then, we have:

$$\vec{a}_1 = -\frac{y}{x}\vec{a}_2 - \frac{z}{x}\vec{a}_3 \quad (12)$$

Then, the determinant is given by:

$$\det(\vec{a}_1, \vec{a}_2, \vec{a}_3) = \det\left(-\frac{y}{x}\vec{a}_2 - \frac{z}{x}\vec{a}_3, \vec{a}_2, \vec{a}_3\right) \quad (13)$$

$$= -\frac{y}{x}\det(\vec{a}_2, \vec{a}_2, \vec{a}_3) - \frac{z}{x}\det(\vec{a}_3, \vec{a}_2, \vec{a}_3) = 0 \quad (14)$$

where in the last step we used the fact that the determinant is zero if the same vector is repeated. Therefore, the proof is complete. I want to note that this proof can be easily generalized to the case of the bigger sized equations.

Now, let us give you an explicit formula for the determinant. For a 3×3 matrix A , it is given by the following formula:

$$\det A = \sum_{i,j,k} \epsilon_{ijk} A_{i1} A_{j2} A_{k3} \quad (15)$$

where ϵ is Levi-Civita symbol For a $n \times n$ matrix A , the above formula can be generalized to the following:

$$\det A = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1 i_2 \dots i_n} A_{i_1 1} A_{i_2 2} \dots A_{i_n n} \quad (16)$$

Problem 1. Show that (16) satisfies the three properties of the determinant. (Hint¹)

Now, we would like to hopefully convince you that determinant of the transpose of a matrix is equal to the determinant of the original matrix. We will consider a 3×3 matrix as an example, and will hope that this example convinces you for matrices with higher size. First, from (15), we have:

$$\det A^T = \sum_{i,j,k} \epsilon_{ijk} A_{1i} A_{2j} A_{3k} \quad (17)$$

Here, you can easily see that each contribution to the sum is zero unless ϵ_{ijk} is non-zero. In other words, i, j and k s have to be some permutations of 1, 2, and 3. That's what the second indices of A s have to be in the above expression; 1, 2, and 3 must appear just once, respectively, neither more nor less. Now, observe also that that's what the first indices of A s are: 1 in A_{1i} , 2 in A_{2j} and 3 in A_{3k} . In other words, both within the first indices and within the second indices 1, 2, and 3 appear just once. Similarly, same can be said about (15). Therefore, if the Levi-Civita symbols that one multiplies to three A s are same, $\det A^T$ is equal to $\det A$. That they are same, we won't justify, but it's intuitively clear with some examples as follows. (Here, we have $\epsilon_{231} = \epsilon_{312}$)

$$\epsilon_{231} A_{12} A_{23} A_{31} = \epsilon_{312} A_{31} A_{12} A_{23} \quad (18)$$

¹The proof of antisymmetry requires the property of Levi-Civita symbol.

Hopefully, this convinces the readers. Of course, it is not a rigorous proof as we didn't justify the last step.

Problem 2. Using (16), show the following:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{21}A_{12} \quad (19)$$

which agrees with the result obtained in our earlier article "Determinant of 2×2 matrices."

Problem 3. Using (16), show the following:

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \quad (20)$$

Problem 4. Explain why the rank of an $n \times n$ matrix can't be n if its determinant is zero.

Problem 5. Let I be an $n \times n$ identity matrix. Then, what is $\det(-I)$?

Summary

- Determinant is a map that sends n n -dimensional vectors to a number.
- Determinant satisfies linearity.
- Determinant satisfies anti-symmetry. i.e. Upon an exchange of two vectors, it picks up a negative sign.
- Determinant of an identity matrix is 1.
- The anti-symmetry of determinant implies that the determinant is zero, if a same vector is repeated in the entry.
- The determinant of a set of vectors that are linearly dependent are zero.