Diagonalization

Suppose you have the following 3 eigenvectors e_1, e_2, e_3 with the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ for an 3×3 matrix A.

$$Ae_i = \lambda e_i \tag{1}$$

Then, we can express this as:

$$A\,(e_1, e_2, e_3) = (e_1, e_2, e_3)\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{2}$$

For example, if $e_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $e_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -2$

We have:

$$A\begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3}$$

Now, let $P = (e1, e2, e3)^{-1}$, then we can write the above equation as:

$$AP^{-1} = P^{-1}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{4}$$

Then, we have:

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{5}$$

So, if we define $A' = PAP^{-1}$, A' is just a similarity transformation of A. Notice also that all the non-diagonal elements of this matrix is zero, and the diagonal elements are given by the eigenvalues. In other words, we changed the basis in such a way that the matrix is maximally simplified. This procedure is called "diagonalization."

Let's take another look at what we have done.

In the new basis,

$$e_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6}$$

and,

$$A' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix}, \quad A' \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix}, \quad A' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix} \tag{7}$$

which is equivalent to

$$A' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{8}$$

implies

$$A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{9}$$

In other words, the eigenvectors are basis as you can see from (6).

Having diagonalized the matrix A in this way, it is easy to see the followings:

$$\det A = \det(PAP^{-1}) = \lambda_1\lambda_2\lambda_3 \tag{10}$$

$$\mathrm{tr}A = \mathrm{tr}(PAP^{-1}) = \lambda_1 + \lambda_2 + \lambda_3 \tag{11}$$

In other words, the above two equations are satisfied for any 3×3 matrix A since its determinant and trace are same as those of its diagonalized ones. In this article, we have considered 3×3 matrix, but all our constructions here can be easily generalized to $n \times n$ matrix. For example, the product of all eigenvalues is the determinant, and the sum of all eigenvalues is trace.

Problem 1. Assuming that a matrix M is diagonalizable, prove the following:

$$\mathrm{Tr} \ln M = \ln \det M \tag{12}$$

Problem 2. Calculate A'^{-1} for (9). What happens when at least one of the eigenvalues is zero (i.e. the determinant (=the product of all eigenvalues) is zero)? Why can't we have A'^{-1} in such a case? On the other hand, show that if the determinant (i.e. the product of all eigenvalues) is non-zero, we can have A'^{-1}.

Summary

- For any generic linear operator, it is possible to find a set of basis, on which the matrix is diagonal.

- Such a process is called diagonalization. One can do this by a similarity transformation.
- The sum of all eigenvalues is trace, and the product of all eigenvalues is determinant.