Differential forms, vector calculus, and generalized Stoke's theorem

There are two ways to express formulas in vector calculus. The first one is using coordinates and the other is using vector notation, which is free of coordinates. For example, using the first way, the dot product of two vectors can be expressed as $v^i w_i$ (using the Einstein summation convention), while, using the second way, the same quantity can be expressed as $\vec{v} \cdot \vec{w}$. Notice that in some respects the coordinate-free method is more advantageous than the one using coordinates, since it is more intuitive; one can lose intuition if one sees too many indices. On the other hand, the expressions using coordinates are more advantageous in explicit calculations.

It is well-known that the gradient, curl and divergence can also be expressed in both ways. However, in expressing vector calculus in a component-free way, one encounters problems if the dimension of the vectors concerned is greater than three. Differential forms provide one method for writing down higher dimensional expressions in a compact, component-free way. Moreover, they allow us to express succinctly Stokes' theorem generalized to any number of dimensions and to understand the origin of Maxwell's equations concretely. Therefore, it's worthwhile to study them. In this text, we review differential forms in enough detail to write down the generalized form of Stokes' theorem. Another easily approachable review of this subject can be found on pages 69 to 76 of Quantum Field Theory by Ryder.

Mathematically speaking, differential forms are objects that can be integrated over a hypersurface to give a number. For example, an n-form is an object that can be integrated over an n-dimensional hypersurface to give a number.

A 1-form is an object that can be integrated over a 1-dimensional line to give out a number. For example, a 1-form w can be represented as follows:

$$w = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz. \tag{1}$$

This makes sense because, as a 1-form, w can be integrated over a one-dimensional line as follows:

$$\int_s w = \int_s f dx + g dy + h dz \tag{2}$$

where s denotes a one-dimensional path.

Then, what would be a 2-form? One might easily suppose that a 2-form, u, can be represented as follows:

$$u = a(x,y,z)dxdy + b(x,y,z)dydz + c(x,y,z)dzdx. \tag{3}$$

This is almost correct, as a 2-form can be integrated over a two-dimensional surface to produce a number. As a two-dimensional surface is two-dimensional we need to integrate twice, so there should be two differentials, such as dx and dy. However, as I stated, this is almost correct rather than being entirely correct, and there is one more complication to expressing higher differential forms that follows - we have to insert $\wedge$ between two differentials such as dx and dy. (We will soon see that the insertion gives the correct coordinate transformation, thus justifying its use.) For example, u above must be re-defined as follows:

$$u = a(x, y, z)dx \wedge dy + b(x, y, z)dy \wedge dz + c(x, y, z)dz \wedge dx. \tag{4}$$

$\wedge$ is called a "wedge product" and has the following property: $dx^i \wedge dx^j = -dx^j \wedge dx^i$. One corollary that immediately follows is that the wedge product of something with itself is equal to zero ($v \wedge v = -v \wedge v \rightarrow v \wedge v = 0$). Wedge products like $dx \wedge dy$ can mostly be thought of acting like $dxdy$ inside of an integrand.

Wedge products also contain information about the orientation of the integration surface. To see this, first notice that the flux integral in vector calculus is integration over a surface. The larger the surface being integrated, the larger the value ($\int \vec{v} \cdot d\vec{A}$). Notice also that in three dimensions, the flux integral is done by taking the dot product of the vector field $\vec{v}$ with the normal vector to the plane $\vec{A}$. We know that we can choose either one of the two normal vectors to the plane; if the plane is given by $z = 0$, the two normal vectors correspond to $\hat{z}$ and $-\hat{z}$. Different choices of the normal vectors give the same magnitude but with opposite signs. The choice of $\hat{z}$ corresponds to the choice of $dx \wedge dy$ in the area integration, while the choice of $-\hat{z}$ corresponds to the choice of $dy \wedge dx$, which is equal to $-dx \wedge dy$. Therefore, the anti-symmetric nature of the wedge product reflects the freedom to choose the orientation of the integration surface (i.e. the normal vector).

Wedge products are also distributive. For example,

$$\begin{aligned}(fdx + gdy) \wedge (hdx + idy) &= fhdx \wedge dx + ghdy \wedge dx + fidx \wedge dy + gidy \wedge dy \\ &= (fi - gh)dx \wedge dy \end{aligned} \tag{5}$$

Notice that (5) gives the correct area element for the parallelogram spanned by two vectors (f, g) and (h, i). Therefore, the use of the wedge product gives the expected Jacobian factor of an integrand under change of variables. In other words, if $da = fdx + gdy$, and $db = hdx + idy$, where $f = \frac{\partial a}{\partial x}$, $g = \frac{\partial a}{\partial y}$, $h = \frac{\partial b}{\partial x}$, and $i = \frac{\partial b}{\partial y}$, we have:

$$\int da \wedge db = \int (fi - gh)dx \wedge dy \tag{6}$$

without the wedge product, it would have been $\int (fi + gh)dxdy + (fh)dxdx + (gi)dydy$, which is not the correct area element.

In this article, we will use a wedge product without a rigorous mathematical justification or proof. As in the above example, one can just note that this definition gives rise to the correct formulas which we can also derive from vector calculus.

To give you another example, let me illustrate how the 3-form volume element behaves under coordinate transformation. Let t, a 3-form be defined as follows: $t = dx \wedge dy \wedge dz$. So this is a volume element. Now, let's perform a change of coordinates to (x', y', z'). Then we get the following:

$$\begin{aligned} dx &= \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial y'}dy' + \frac{\partial x}{\partial z'}dz' \\ dy &= \frac{\partial y}{\partial x'}dx' + \frac{\partial y}{\partial y'}dy' + \frac{\partial y}{\partial z'}dz' \\ dz &= \frac{\partial z}{\partial x'}dx' + \frac{\partial z}{\partial y'}dy' + \frac{\partial z}{\partial z'}dz' \end{aligned}$$

Now, if we re-express t, reminding ourselves that $dx' \wedge dy' \wedge dz' = dy' \wedge dz' \wedge dx' = dz' \wedge dx' \wedge dy' = -dy' \wedge dx' \wedge dz' = -dz' \wedge dy' \wedge dx' = -dx' \wedge dz' \wedge dy'$, we get the following for the 3-form in terms of new coordinates:

$$t = dx \wedge dy \wedge dz = \det \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{vmatrix} dx' \wedge dy' \wedge dz' \tag{7}$$

So, the antisymmetric nature of the wedge product gives rise to the expected Jacobian factor of an integrand under change of variables.

Having said this, we have a concrete basis on which to introduce the exterior derivative, "d", which is a generalization of the gradient, curl, and divergence. The exterior derivative of a k-form is a $(k+1)$-form. For example, an exterior derivative of a 0-form (one with no dxs or dys) is a 1-form. We can write this as follows:

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz \tag{8}$$

where A is a 0-form. Surprisingly, this is a gradient. Notice that the exterior derivative is a coordinate-independent quantity, just as the dot product is. For example, the above equation can be expressed in another coordinate system as follows:

$$dA = \frac{\partial A}{\partial x'}dx' + \frac{\partial A}{\partial y'}dy' + \frac{\partial A}{\partial z'}dz' \tag{9}$$

which is the same vector as in (8), but just in another coordinate system.

We can extend the definition of the exterior derivative to a general n-form by the following rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha}(\alpha \wedge d\beta) \tag{10}$$

where $\deg \alpha$ denotes the degree of α. The degree of an n-form is defined to be n. The term $(-1)^{\deg \alpha}$ takes care of the antisymmetric nature of the wedge product.

Moreover, the exterior derivative has the property that $d^2 = 0$. One corollary that immediately follows from this is that the exterior derivative of dx is zero. In other words $d(dx) = 0$. Similarly, we have $d(dy) = d(dz) = 0$.

Now, let's see what the exterior derivative of a 1-form is. Let α be a 1-form defined by $\alpha = V_x dx + V_y dy + V_z dz$. Then we get the following:

$$d\alpha = dV_x dx + V_x d(dx) + dV_y dy + V_y d(dy) + dV_z dz + Vzd(dz) = dV_x dx + dV_y dy + dV_z dz \quad (11)$$

So, you can see here that V_x, V_y, and V_z are 0-forms and we can use formula (8) to replace dV_x, dV_y, dV_z. Then we get

$$\begin{aligned}
d\alpha &= (\frac{\partial V_x}{\partial x}dx + \frac{\partial V_x}{\partial y}dy + \frac{\partial V_x}{\partial z}dz) \wedge dx + (\frac{\partial V_y}{\partial x}dx + \frac{\partial V_y}{\partial y}dy + \frac{\partial V_y}{\partial z}dz) \wedge dy \\
&+ (\frac{\partial V_z}{\partial x}dx + \frac{\partial V_z}{\partial y}dy + \frac{\partial V_z}{\partial z}dz) \wedge dz \\
&= (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z})dy \wedge dz + (\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x})dz \wedge dx + (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})dx \wedge dy
\end{aligned}$$

If one defines a vector $U = U_x dy \wedge dz + U_y dz \wedge dx + U_z dx \wedge dy$, surprisingly, we get $\nabla \times \vec{\alpha} = \vec{U}$.

Similarly, let's see what the exterior derivative of a 2-form is. Let $U = U_x dy \wedge dz + U_y dz \wedge dx + U_z dx \wedge dy$ be a 2-form. Then we get the following:

$$\begin{aligned}
dU &= \frac{\partial U_x}{\partial x}dx \wedge dy \wedge dz + \frac{\partial U_y}{\partial y}dy \wedge dz \wedge dx + \frac{\partial U_z}{\partial z}dz \wedge dx \wedge dy \\
&= (\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z})dx \wedge dy \wedge dz \quad (12)
\end{aligned}$$

You immediately see that this is the divergence.

Summarizing, the exterior derivative of a 0-form is the gradient, that of a 1-form is the curl, and that of a 2-form is the divergence. Now, let's consider why d^2 is zero. This is so because partial derivatives commute. In other words, when we calculate d^2 for an n-form, we unavoidably have terms such as $\frac{\partial^2 f}{\partial x \partial y} dx \wedge dy$, which are always canceled by terms such as $\frac{\partial^2 f}{\partial y \partial x} dy \wedge dx$.

Having said this, d^2=0 implies that the curl of a gradient is zero and that the divergence of a curl is zero. Moreover, the divergence theorem or Green's theorem can be generalized to Stokes' theorem in any dimensionality, which states that $\int_{\partial S} \omega = \int_S d\omega$ where ∂S is the boundary of the n-dimensional surface S and $d\omega$ is the exterior derivative of the $(n-1)$-form ω. When ω is a 0-form and S is a one-dimensional object, we recover the fundamental theorem of calculus as follows: $\int_a^b f(x)dx = F(b) - F(a)$, where $\frac{dF(x)}{dx} = f(x)$. When ω is a 1-form and S is a two-dimensional object, we recover Green's theorem. When ω is a 2-form and S is a three-dimensional object we recover the divergence theorem. In the next article, we will express Maxwell's equations in terms of differential forms, showing the benefit of this formalism. Personally, after encountering this problem as a homework in my freshman honors mathematics class, I was mesmerized and convinced that a simple "Theory of Everything" must exist, as Maxwell's equations are greatly simplified when one uses differential forms.