Electric potential and vector potential

If magnetic field can be written as a curl of a vector, the condition that the divergence of magnetic field is zero can be always satisfied since the divergence of the curl of any vector is always zero. In fact, it is also known that for any divergence-less vector, there exists other vector whose curl is the divergence-less vector. Therefore, for some $\vec{A}$, we can write the magnetic field $\vec{B}$ as follows:

$$\vec{B} = \nabla \times \vec{A} \tag{1}$$

This is called "vector potential." Similarly, it is known that an electric field $\vec{E}$ can be written in terms of electric potential ϕ and the same vector potential $\vec{A}$ as follows:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

This satisfies one of other Maxwell's equations since:

$$\nabla \times \vec{E} = -\nabla \times \nabla\phi - \frac{\partial(\nabla \times \vec{A})}{\partial t} = -\frac{\partial B}{\partial t} \tag{3}$$

The satisfaction of other Maxwell's equations is more subtle. Actually, there aren't unique $\vec{A}$ and ϕ choices that satisfy Maxwell's equations such as (1) and (2). In other words, we have choices. This is called "gauge choice," and choosing such one is called "choosing a gauge." We will return to this point later in other articles.

Problem 1. Show that the magnetic flux through a closed loop is equal to the line integral of vector potential along the loop. In other words,

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \oint \vec{A} \cdot d\vec{s} \tag{4}$$

(The notation may be a little bit confusing. $\vec{A}$ in $d\vec{A}$ is the area element, while $\vec{A}$ in $\vec{A} \cdot d\vec{s}$ is the vector potential.)

Summary

- The vector potential $\vec{A}$ is defined to satisfy $\vec{B} = \nabla \times \vec{A}$.
- $\vec{E} = -\nabla\phi - \dfrac{\partial \vec{A}}{\partial t}$.
- The magnetic flux can be expressed using the vector potential as follows

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \oint \vec{A} \cdot d\vec{s}$$