Electric dipole

See Fig.1. We have two charges, $+q$ and $-q$ separated by a distance d, aligned along y-axis. In the middle is the origin O. In this article, we will obtain the electric potential, and the electric field due to the two charges altogether, at point A which is distance $r >> d$ away from the origin, making an angle θ with x-axis. In other words, the location of A is given by (r, θ) in polar coordinate. In the figure, the distance between A and the charge $+q$ is denoted as r_+ while the distance between A and the charge $-q$ is denoted as r_-. Both r_+ and r_- are very close to r as $r >> d$, and their difference is given by $r_- - r_+ \approx d \sin\theta$. (If you are not sure how this is derived, remember how the path difference was obtained in our earlier article "Young's interference experiment." It's exactly the same.) Given this, now let's calculate the electric potential at A. We have:

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) = \frac{q}{4\pi\epsilon_0}\frac{r_- - r_+}{r_- r_+} \approx \frac{q}{4\pi\epsilon_0}\frac{d\sin\theta}{r^2} = \frac{qd}{4\pi\epsilon}\frac{\sin\theta}{r^2} \tag{1}$$

We see here that apart from the location the electric potential only depends on the combination qd instead of q and d itself. In other words, as long as $r >> d$ is satisfied, we wouldn't be able to find out q and d separately but only the combination qd from measuring the electric field. For example, if we double q and halve d, the electric field will remain the same. All this suggests that it will be useful to give a name to the combination qd. It is called "electric dipole moment" and usually denoted by p. (i.e. $p = qd$) Notice also that the potential is inversely proportional to r^2 rather than r as is the case with a single charge. We can roughly interpret this as the electric potential of positive charge and the one of the negative charge canceling each other but not completely, making the "left-over" electric field quite small.

Now, let's find the electric field. Remember $\vec{E} = -\nabla V$. Then, using the following formula which you proved in "Kinetic energy and Potential energy in polar coordinate,"

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} \tag{2}$$

you get:

$$\vec{E} = \frac{p}{4\pi\epsilon_0}\frac{2\sin\theta}{r^3}\hat{r} - \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^3}\hat{\theta} \tag{3}$$

You see here that the electric field is inversely proportional to r^3 rather than r^2 as is the case with single charge. As was the case with electric potential, we can interpret this as electric field of the positive charge and the one of the negative charge canceling each other but not completely.

The expressions (1) and (3) we obtained are not coordinate-free; we have arbitrarily chosen y-axis as the line the two charges are aligned. Is there any way to re-express the two

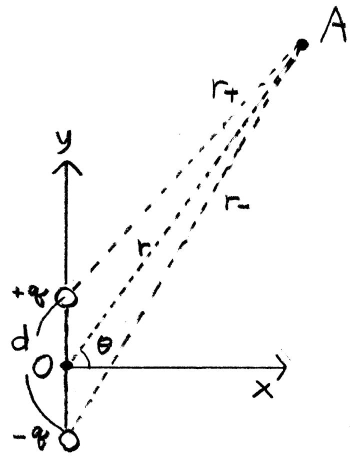

Figure 1: electric potential and electric field of an electric dipole

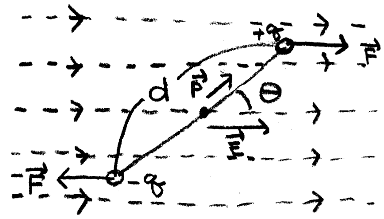

Figure 2: electric dipole in a constant electric field

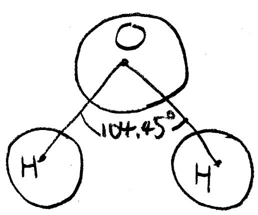

Figure 3: water molecule

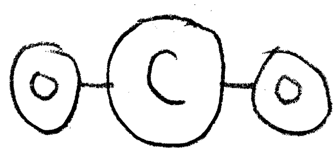

Figure 4: molecule of carbon dioxide

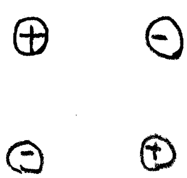

Figure 5: electric quadrupole

expressions in a coordinate-free way? The answer is yes, if we upgrade the electric dipole moment to a vector by giving it a direction, and re-express the two expressions in terms of it. By convention, the direction is given by the vector from the negative charge to the positive charge, which in our case is given by $\hat{j}$. Therefore, in our case, the electric dipole moment vector is given by $\vec{p} = qd\hat{j}$.

Then, remembering $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ from our earlier article "Kinetic energy and Potential energy in polar coordinate," we can re-write (1) as

$$V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \tag{4}$$

while (3) can be re-expressed as (**Problem 1.** Check! Hint[1])

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} \tag{5}$$

Now, we will consider an electric dipole in a constant electric field. See Fig.2. We will assume that the two charges are bonded each other so their relative distance is fixed. The dotted lines denote electric field. You also see that the electric dipole moment makes an angle θ with the electric field. The positive charge receives the force $\vec{F} = q\vec{E}$ which is rightward, while the negative charge receives the force $-\vec{F} = -q\vec{E}$ which is leftward. Therefore, the total force is zero, so there is no net force, which makes the center of mass (i.e. the right middle point of two charges denoted in Fig.2. by a black dot) remain there. However, the dipole receives a torque; it rotates around the center of mass, which is the fixed axis. The electric force on the positive charge contributes $(d/2)F\sin\theta$ to the torque and the one on the negative charge contributes to the same amount as well. (Both of their contribution to the torque is in the same direction, being clock-wise.) Therefore, the total torque is:

$$\tau = (d/2)F\sin\theta \times 2 = dF\sin\theta = dqE\sin\theta = pE\sin\theta \tag{6}$$

Using the cross product, the above formula can be re-expressed as

$$\tau = \vec{p} \times \vec{E} \tag{7}$$

Now, let's assume that the dipole was at an arbitrary initial angle $\theta = \theta_0 \neq 0$. Then, it will begin to rotate. If θ_0 is positive, clock-wise, if it is negative, anti-clock-wise. In any case, the rotating speed will get bigger and bigger until it reaches $\theta = 0$, at which point the rotating speed is maximum. Beyond this point, the dipole will keep rotating but the direction of the rotation will be the direct opposite of the torque, which makes the rotating speed smaller and smaller. Eventually, it will reach the angle $\theta = -\theta_0$ and will momentarily stop. Then, it will rotate in the other direction. So, this will continue on and on. This is exactly like pendulum.

Problem 2. Calculate the period of this electric dipole "pendulum" assuming that the amplitude (i.e. θ_0) is very small and the mass of each charge is m. (In January 1996, when

[1] Show and use $\hat{j} = \sin\theta\hat{r} + \cos\theta\hat{\theta}$

I studied college physics for the first time at physics olympiad camp, I asked an older friend how to solve this problem, as I didn't know how. He answered, but I still didn't understand his explanation.)

The electric dipole "pendulum" has a potential energy, as much as a usual pendulum has a potential energy. The usual pendulum has minimum potential energy when $\theta = 0$ as it is the lowest point in the orbit. Also, the higher potential energy the bigger θ. Same can be said for the electric dipole "pendulum." The rotating speed is fast at the point $\theta = 0$ as it has the minimum potential energy at this point.

Of course, we can calculate the potential energy. The rotational analog of $U = -\int F \cdot ds$ is given by $U = -\int \tau d\theta$. Considering that τ is in the direction in which θ is decreasing when θ, as well as $\sin\theta$ is positive, (6) must be re-written as $\tau = -pE\sin\theta$ with a negative sign, strictly speaking. This yields:

$$U = -\int(-pE\sin\theta)d\theta = -pE\cos\theta = -\vec{p}\cdot\vec{E} \tag{8}$$

Let me conclude this article with two comments. First, electric dipole exists in our nature. A molecule has a non-zero electric dipole moment, if the average position of positive charges doesn't coincide with that of negative charge. For examples, see Fig.3. and Fig.4. A water molecule, which consists of two hydrogen atoms and one oxygen atom has non-zero electric dipole moment, while a carbon dioxide molecule which consists of one carbon atom and two oxygen atoms has no electric dipole moment.

Second, there is something called "electric quadrupole." See Fig.5. Two dipoles separated by a small distance are aligned in opposite directions. The electric potential is inversely proportional to r^2 and the electric field r^3. There is also something called "electric octupole." Positive electric charges and negative electric charges are placed at the eight corners of a cube alternatively. Of course, the electric potential is inversely proportional to r^3 and the electric field r^4.

Problem 3. Consider a case in which the electric field is not constant, but depends on the position. Then, without using $U = -\vec{p}\cdot\vec{E}$, show that the electric dipole receives a force $\vec{F} = \vec{p}\cdot\nabla\vec{E}$. (That it receives a non-zero force is natural as the electric field at the location of the positive charge and the one at the location of the negative charge are different.)

Summary

- If charges q and $-q$ are located very close to each other, the electric dipole moment is a vector from the location of the negative charge to the location of the positive charge with magnitude qd where d is the distance between them.
- The electric potential due to an electric dipole is inversely proportional to r^2, and the electric field is inversely proportional to r^3.
- If there is an external electric field $\vec{E}$, the electric dipole receives the torque $\tau = \vec{p}\times\vec{E}$, and its potential energy is given by $U = -\vec{p}\cdot\vec{E}$.