The expanding universe

As mentioned in our earlier article "History of astronomy," Hubble found out that our universe was expanding; the farther the galaxies, the faster they were moving away from us. As mentioned in our earlier article "The Doppler effect and the twin paradox revisited," we know their receding speed from the Doppler effect. In this article, we will explain again all this more mathematically, and in more detail.

Suppose that, currently (let's say time $t = t_0$), galaxy A and galaxy B are separated by 100 million light years, and galaxy A and galaxy C are separated by 300 million light years. Let's say the universe continues to expand and it doubles its size at time $t = t_d$. Then, the distance between galaxy A and galaxy B will be doubled (i.e., 200 million light years), and as well as the distance between galaxy A and galaxy C (i.e., 600 million light years).

Now, let us introduce the "scale factor" a. The scale factor is defined to be proportional to the size of the universe. As the size of universe at $t = t_d$ is the double the size of the universe now, we have $a(t_d) = 2a(t_0)$. In the literature, we often denote the scale factor of the current time as $a_0 \equiv a(t_0)$ and normalize it to be $a_0 = 1$. However, we denote both the scale factor a hundred years ago, and the scale factor now as a_0. You may worry that this may bring some confusion, as the scale factor now and then are different, as the universe expanded last 100 years, but this doesn't bring any confusion, because the scale factor hasn't changed noticeably as 100 years is a very short period compared to the age of the universe which is in the order of 10 billion years.

Using this convention, we have $a(t_d) = 2$. Now, if we re-calculate the distance between galaxy A and galaxy B at $a(t_d) = 2$, we can simply put

$$a(t_d) \times 100 \text{ million light years} = 200 \text{ million light years} \tag{1}$$

Similarly, the distance between galaxy A and galaxy C at $a(t_d) = 2$ is given by

$$a(t_d) \times 300 \text{ million light years} = 600 \text{ million light years} \tag{2}$$

As the distance between galaxies are changing, it is sometimes useful to introduce a distance that *doesn't* change as the universe is expanding. This is called "co-moving" distance. It is easy to see that, if we divide the actual distance by the scale factor, we will get a co-moving distance. For example, the co-moving distance between A and B is given by 100 million light years, as

$$\frac{100 \text{ million light years}}{a(t_0)} = \frac{200 \text{ million light years}}{a(t_d)} = 100 \text{ million light years} \tag{3}$$

You indeed see that the co-moving distance now is the same as the co-moving distance when $t = t_d$. Thus, if we denote the actual distance as $d(t)$, and the co-moving distance as x, we have $x = d(t)/a(t)$. In other words,

$$d(t) = a(t)x \tag{4}$$

Now, let's say that we are on the Earth, and consider a galaxy x co-moving distance away from us. How fast are they moving away from us? It is given by

$$\dot{d}(t) = \dot{a}(t)x \tag{5}$$

If we express the above quantity using the actual distance (often called the "proper distance") $d(t)$, we have

$$\dot{d}(t) = \frac{\dot{a}(t)}{a(t)} d(t) \tag{6}$$

Now, if we define the Hubble constant $H(t)$ as follows

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{7}$$

then, we have

$$\dot{d}(t) = H(t)d(t) \tag{8}$$

This is known as Hubble's law. The farther away the galaxy is from us (the bigger d), the faster they are moving away from us (the bigger $\dot{d}$).

Of course, the Hubble constant is not a genuine constant as it depends on time, but it is a constant in the sense that it is the proportionality constant for the speed of galaxy to the distance to the galaxy. Notice also that the Hubble constant doesn't depend on the normalization of $a(t)$; if we defined $a(t_0) = 3$ instead of $a(t_0) = 1$, $\dot{a}(t_0)$ would be triple of the one in which we defined $a(t_0) = 1$. This shows that $\dot{a}/a$ doesn't change under such re-definition. Another way of seeing this is that the Hubble constant can be defined without explicitly referring to $a(t)$, if we write (8) as

$$H(t) = \frac{\dot{d}(t)}{d(t)} \tag{9}$$

Let's now define the red-shift z. If the wavelength emitted with wavelength λ_{em} is observed to be λ_{ob} on the Earth, the red-shift is given by

$$z = \frac{\lambda_{\text{ob}}}{\lambda_{\text{em}}} - 1 \tag{10}$$

For example, if the wavelength increases by 10%, then the red-shift is 0.1.

Problem 1. Consider a galaxy d distance away from us. Assuming that this galaxy's receding speed is not big compared to the speed of light c, show that the red-shift is given by

$$z = \frac{dH}{c} \tag{11}$$

There is an equivalent, but more correct way of understanding this red-shift. When light was emitted from a far away galaxy, the universe was smaller than now. In other words, the scale factor was smaller. However, as the universe expands, the wavelength was lengthened in tandem by the same ratio. (If you can't believe this, read our later article "Horizons.") Therefore, we can write

$$\frac{\lambda_{\rm ob}}{\lambda_{\rm em}} = \frac{a_0}{a} \tag{12}$$

where a denotes the scale factor of the universe when the light was emitted. Therefore, we conclude

$$a = \frac{a_0}{1+z} \tag{13}$$

We can actually show that this equation leads to (11). It takes the time d/c for light from a galaxy distance d to reach us. Then, the scale factor then was given by

$$a(t_0 - \frac{d}{c}) \approx a(t_0) - \frac{d}{c}\dot{a}(t_0) \tag{14}$$

Problem 2. Finish the calculation to derive (11).

So, why is the universe expanding now? It is because the universe was expanding in the past. If there is no reason to stop it, it should continue to expand. However, when naively thought, there may be reasons to think that the expansion of the universe should slow down. In the universe, there are a lot of matters, and from Newton's law we know that matters attract each other. Thus, the matters in the universe should slow down the expansion of the universe. This analysis is still valid in Einstein's general relativity which is a theory of gravity.

However, as mentioned in our earlier article, observations show that the universe is currently expanding faster and faster. We say our universe is "accelerating." This was concluded by measuring the deviation from Hubble's law (11); roughly speaking, if the universe is accelerating, the expansion rate of the universe H was smaller in the past, which implies galaxies farther away, which emitted light longer ago, to reach our telescope now, is not moving away from us as fast as it should if H then were same as now.

This implies that, in our universe, besides ordinary matter such as galaxies or light that would slow down the expansion of the universe, there are other new unknown contents. They are called "dark energy." We do not know much about the nature or the origin of dark energy.

Summary

- The universe expands. The size of the universe and the distances between galaxies are proportional to the "scale factor" $a(t)$.
- The scale factor now is often denoted as a_0 and often set to be 1.
- The Hubble constant H is given by

$$H = \frac{\dot{a}}{a}$$

- For a galaxy d distance away from us, the receding speed is given by

$$v = Hd$$

- The farther the galaxies are, the more red-shifted the light from them, as they are receding faster.

- The red-shift is defined by

$$z = \frac{\lambda_{\rm ob}}{\lambda_{\rm em}} - 1$$

- An equivalent, but more correct way of seeing red-shift is that the wavelength is stretched as the size of universe grows. Thus,

$$a = \frac{a_0}{1+z}$$

  where a is the scale factor when the light was emitted.