# Forced harmonic oscillator

In this article, we will consider two examples of differential equations. The first one is damped harmonic oscillator and the second one is the forced harmonic oscillator. To solve the latter, you need to solve the former.

## 1 Damped harmonic oscillator

In a simple harmonic oscillator the spring oscillates forever. However, in everyday life, the oscillation stops because of the damping due to air. Also, it is easy to imagine that the force due to damping is bigger when the spring moves faster, and the damping force is exerted in the opposite direction of the movement. These considerations suggest the following differential equation.

$$m\ddot{x} = -kx - b\dot{x} \tag{1}$$

where the second term in the right-hand side of the above equation is due ot damping. Now, let's solve this equation. We try the following solution:

$$x = Ae^{ct} \tag{2}$$

Then, we have:

$$mc^2 + cb + k = 0 \tag{3}$$

(Problem 1. Check this.) which yields:

$$c = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{4}$$

Now, if $b^2 < 4mk$, we have:

$$c = -\frac{b}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}i \tag{5}$$

Plugging this back into (2), we get:

$$x = Ae^{-bt/2m}\cos(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t) + iA\sin(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t) \tag{6}$$

We now need to notice the following. If $x$ is a solution to (1), its real part as well as its imaginary part is also a solution. Therefore, if we say $A = D + Ei$, the real part of the above equation is

$$De^{-bt/2m}\cos(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t) - E\sin(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t) \tag{7}$$

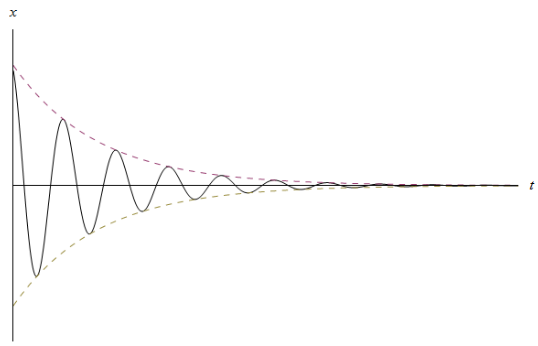


Figure 1: Damped harmonic oscillator

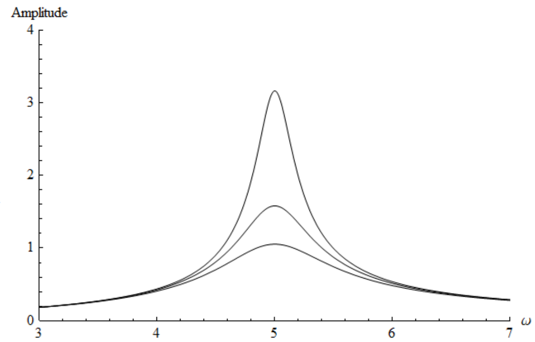


Figure 2: Resonance in forced harmonic oscillator

Summarizing, the real solution to (1) can be written as

$$x = x_m e^{-bt/2m} \cos(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} t + \phi_0) \tag{8}$$

Now, we need some interpretation. The cosine part shows that this is oscillating, and the $x_m e^{-bt/2m}$ is the amplitude of the oscillation. Notice that the amplitude decays exponentially as time goes on. See Fig.1. The dotted line denotes the amplitude while the solid line the actual $x$. On the other hand, what would happen if $b^2 \geq 4mk$? This is the case in which the damping force is too big to oscillate even once. We can see that sine or cosine functions are absent from the solution to the differential equation, as $c$ doesn't have any imaginary part.

## 2 Forced harmonic oscillator

To keep the spring oscillate, you need an external force. Let's say that the external force is given by $F_0 \sin(\omega t)$. Then, we have:

$$m\ddot{x} = -kx - b\dot{x} + F_0 \sin(\omega t) \tag{9}$$

We can solve this differential equation guessing the following form of solution:

$$x(t) = x_0 \sin(\omega t + \phi) \tag{10}$$

Plugging this in, we get:

$$x_0 = \frac{F_0}{mZ\omega}, \qquad \phi = \tan^{-1}\left(\frac{m}{b}\sqrt{\frac{m}{k}}(\frac{k}{m} - \omega^2)\right) \tag{11}$$

where

$$Z = \sqrt{(\frac{b}{m})^2 + \frac{1}{\omega^2}(\omega^2 - \frac{k}{m})^2} \tag{12}$$

Given this, what is the velocity? We can simply differentiate (10) with respect to $t$. We obtain:

$$v(t) = \frac{F_0}{mZ} \cos(\omega t + \phi) \tag{13}$$

So the amplitude of the velocity is given by $F_0/(mZ)$. Now, when is this greatest? It is when $Z$ is smallest. This is when the second term inside the square root in (12) is zero, since this term is smallest when it is zero. Therefore, the amplitude of the velocity is greatest when $\omega = \sqrt{k/m}$.

Now notice that this is also the value for $\omega$ when the spring would oscillate in the absence of damping and external force. In other words, the spring oscillates with maximum amplitude, when external force with "the natural frequency" of the spring is exerted. See Fig.2. We have drawn the velocity amplitude of forced harmonic oscillator in terms of $\omega$ for three different values of $b$. Here, we have set $\sqrt{\frac{k}{m}} = 5$. We see that there is a peak for this value of $\omega$. The phenomenon that the spring oscillate rapidly in the certain frequency is called "resonance." The narrowest and the tallest graph corresponds to $b = 1$, and the middle one $b = 2$, and the widest and the shortest graph corresponds to $b = 3$. We see that the bigger the damping the less the amplitude which sounds reasonable. Also, we could have considered the amplitude for the displacement instead of the amplitude for the velocity, but the maximum amplitude for the displacement is not achieved when $\omega = \sqrt{\frac{k}{m}}$ but slightly different value. In any case, it is achieved when it is "about" $\omega = \sqrt{\frac{k}{m}}$.

I conclude this article with three comments. Strictly speaking, the solution to (9) which we obtained in this section is not unique. Let's re-write the differential equation here again for convenience.

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin(\omega t) \tag{14}$$

Notice that the solution to the above equation can be written as follows:

$$x = x_h + x_p \tag{15}$$

Here, $x_h$ is called homogeneous solution, and satisfies following:

$$m\ddot{x}_h + b\dot{x}_h + kx_h = 0 \tag{16}$$

i.e. it is given by the solution to (1) which is (8). And, $x_p$ is called particular solution and satisfies following:

$$m\ddot{x}_p + b\dot{x}_p + kx_p = F_0 \sin(\omega t) \tag{17}$$

i.e. it is given by the solution to (9) which is (10). Notice that the homogeneous solution exponentially decays, which implies that only particular solution survives after enough time has elapsed.

My second comment is that in some electric circuit problems we have the same differential equation as the forced harmonic oscillator considered in this article. Only the coefficients are different. It often happens that apparently different physical phenomena in our universe have the same equation.

My third comment is from *Fundamentals of Physics* by Halliday, Resnick and Walker. There was a big earthquake near Mexico City in 1985. Mexico city was severely damaged.

However, while shorter buildings and taller buildings weren't damaged much, intermediate-height buildings were severely damaged. The reason was that the seismic wave was concentrated around 2 Hz, which happened to be the resonance frequency of intermediate-height buildings. Thus, shorter buildings which have higher resonant frequency and taller buildings which have lower resonant frequency were damaged less.

## Summary

- In a damped harmonic oscillator, there is a damping force in addition to the usual Hooke's force. This damping force damps the oscillator; the amplitude decreases.
- In forced harmonic oscillator, there is an external periodic force to make the damped harmonic oscillator keep oscillating. The amplitude of the velocity is the greatest when the period of the external force matches the natural frequency of the harmonic oscillator. This is known as "resonance."