Geometric series

Suppose you are to calculate the following sum:

$$S = 1 + 0.6 + 0.6^2 + 0.6^3 + 0.6^4 + \cdots + 0.6^{20} \tag{1}$$

Don't you think it would take too long a time to add up everything even with a calculator?

Here, I present a trick to calculate this easily; it is easy to see the following.

$$0.6S = 0.6 + 0.6^2 + 0.6^3 + 0.6^4 + \cdots + 0.6^{21} \tag{2}$$

Subtracting this from (1), we get:

$$(1 - 0.6)S = 1 - 0.6^{21} \tag{3}$$

Therefore, we conclude:

$$S = \frac{1 - 0.6^{21}}{1 - 0.6} \tag{4}$$

This trick makes the calculation much simpler. We have just considered a sum of geometric sequence; a geometric sequence is a sequence that has a constant ratio between consecutive terms.

Actually, the real usefulness of this trick comes when we consider an infinite sum of geometric sequence as follows:

$$T = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \tag{5}$$

Of course, no one can spend an infinite amount of time to sum up infinitely many terms. In the above formula, it is evident that this sum will not converge if x is bigger or equal to 1. The more terms you consider, the bigger the answer becomes. However, if x is smaller than 1 and bigger than -1, it is easy to see that this converges. By using the same trick as our earlier example, we get:

$$T = \frac{1 - x^{\infty}}{1 - x} \tag{6}$$

Since in our case (i.e. $-1 < x < 1$) x^{∞} is zero, we can conclude:

$$T = \frac{1}{1 - x} \tag{7}$$

Problem 1. Show

$$\sum_{m=1}^{\infty} x^m = \frac{x}{1-x} \tag{8}$$

when $-1 < x < 1$. These results are crucial to understand my first published research paper which we will review in our later article "Approximation of the naive black hole degeneracy."

Problem 2. In the 5th century B.C. the Greek philosopher Zeno came up with three paradoxes, now called, Zeno's paradoxes to argue that motion is nothing but an illusion. The first paradox concerns Achilles and the tortoise, the story that we introduced in our article "Linear equations." In the race, the tortoise had a head start of 100 meters, and Achilles's speed is 10 m/s and the tortoise's speed is 1 m/s. When Achilles reached the position where the tortoise started off, the tortoise will be at a new position, certain distance away from the starting point, as the tortoise moves while Achilles runs to the tortoise's original position. So, Achilles needs to run further to get to the tortoise's new position. However, the tortoise doesn't sit and rest while Achilles runs further to tortoise's new position; the tortoise further moves and Achilles needs to catch up to tortoise's newer position again, again and again. Thus, Zeno concluded that Achilles will never catch up the tortoise. Of course, this is a wrong conclusion. While Achilles runs for his first 100 meters, the tortoise moves 10 meters. Then, Achilles further needs to run for 10 meter, during which the tortoise moves 1 meter. Then, once again Achilles needs to run for 1 meter, while the tortoise moves 0.1 meter and so on. Obtain the distances that Achilles needs to run to catch up the tortoise by summing up each distance. You will need to sum up an infinite geometric series. Check that your answer is correct by comparing with the answer we got in our article "Linear equations."

Summary

- A geometric sequence is a sequence that has a constant ratio between consecutive terms.
- A geometric series can be easily calculated by writing out the terms in the first line, and the terms multiplied by the ratio of consecutive terms in the second line (let's call the ratio "r"), and by subtracting the second line from the first line by eliminating the same terms. Then, all but the first term in the first line and the last term in the second line are canceled. Finally, by dividing the result by $(1 - r)$, you can calculate the series.