# Group velocity and phase velocity

As advertised, in this article we will resolve the paradox that the wave function doesn't seem to move at the speed of the object that the wave function is describing. Actually, what we are going to talk doesn't only concern the wave function in quantum mechanics, but also waves in general.

To gain insight, we will first consider a simple case in which a wave is a superposition of two sine waves. Then, we will consider the general case in which the wave is a superposition of many sine waves.

See Fig.1. The solid line is the wave given by

$$\psi = \sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t) \tag{1}$$

where $k_1$ is very close to $k_2$, and $\omega_1$ is very close to $\omega_2$. Given this, using the following formula:

$$\sin A + \sin B = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2} \tag{2}$$

the wave function can be re-expressed as:

$$\psi = 2 \cos \left( \frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t \right) \sin \left( \frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t \right) \tag{3}$$

In other words, $\psi$ can be seen as a wave with "amplitude" $A$, the wave number $k$, and the angular frequency $\omega$ given as follows:

$$A = 2 \cos \left( \frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t \right), \quad k = \frac{k_1 + k_2}{2}, \quad \omega = \frac{\omega_1 + \omega_2}{2} \tag{4}$$

The "amplitude" (i.e. "envelope") is denoted in the figure as dotted line. You see that "ripples" are inside envelope. The speed the ripple travels is called "phase velocity" and given by

$$v_p = \frac{\omega}{k} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \tag{5}$$

And, the speed the envelope travels is called "group velocity" and given by

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta \omega}{\Delta k} = \frac{d\omega}{dk} \tag{6}$$

Both phase velocity and group velocity are denoted in the figure with the arrows. Depending on cases, group velocity can be equal to, greater than, or less than phase velocity. For examples, group velocity is equal to phase velocity for waves on a string, and is half the phase velocity for water waves.

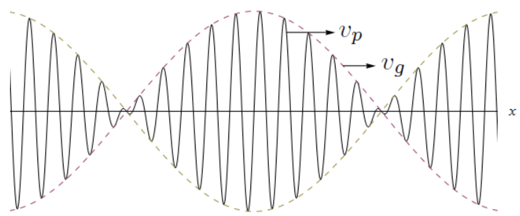


Figure 1: Superposition of two sine waves

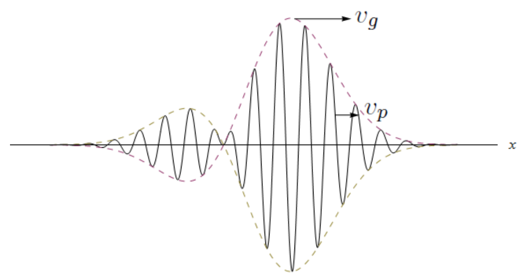


Figure 2: Superposition of many sine waves

The speed the actual wave propagates is group velocity as it is the actual chunk (i.e. "group" or "envelope") moves. If you are not convinced, see Fig.2. The wave there is a superposition of many sine waves. Both the group velocity and the phase velocity are denoted in the figure. You again see that the envelope moves with group velocity while the ripples moves with the phase velocity inside the envelope.

Let's prove this mathematically. First, by Fourier transformation a wave can be expressed as:

$$\psi(x,t) = \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega t)} dk \tag{7}$$

Now, let us assume that the Fourier mode $\phi(k)$ is sharply peaked around $k = k_0$. (Otherwise, there are multiple chunks that move at different speeds as their values for $v_g$ would be different for different values for $k$. Remember that we are deriving a single $v_g$ for a given $k$.) Then, we can write:

$$\omega(k) \approx \omega_0 + \frac{d\omega}{dk}(k - k_0) \tag{8}$$

where $\omega_0 = \omega(k_0)$. Then, we have

$$\psi(x,t) \approx e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \phi(k) e^{i(k-k_0)(x - \frac{d\omega}{dk} t)} dk \tag{9}$$

We see that the first factor $e^{i(k_0 x - \omega_0 t)}$ corresponds to $\sin\left(\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t\right)$ in our earlier example. (i.e. the ripples inside the envelope) And, the integrand is the amplitude or the envelope. We see that it moves at the speed $\frac{d\omega}{dk}$. More rigorously, we see (**Problem 1.** Check! Hint[1])

$$|\psi(x,t)|^2 \approx |\psi(x - \frac{d\omega}{dk}t, 0)|^2 \tag{10}$$

This completes the proof.

Now, let's calculate the group velocity for the wave function in quantum mechanics. We have:

$$v_g = \frac{d\omega}{dk} = \frac{\hbar d\omega}{\hbar dk} = \frac{dE}{dp} = \frac{d\left(p^2/2m\right)}{dp} = \frac{p}{m} \tag{11}$$

Thus, we resolved the paradox which we suggested in our earlier article!

Let us conclude this article with a comment. We said that the Fourier mode of the wave packet is peaked around certain $k$, which we called $k_0$. Nevertheless, it is certainly true that there is not a single mode $k_0$ but other modes around $k_0$. (Otherwise, the particle the wave function describes cannot be localized. Remember our discussion in our earlier article on Heisenberg's uncertainty principle.) Now, notice that the Fourier modes whose wave number are bigger than $k_0$ will move faster than the one with $k_0$ as the former's group velocity is bigger than the latter. Similarly, the Fourier modes whose wave number are smaller than $k_0$ will move slower than the one with $k_0$. Therefore, the wave packet will spread itself as time goes on. This is an important effect when sending signals through optical fiber.

## Summary

- Group velocity, given by $v_g = \frac{d\omega}{dk}$ is the actual velocity the chunk of wave propagates.

[1] Show that $\psi(x,t) \approx e^{i\theta}\psi(x - \frac{d\omega}{dk}t, 0)$ for some pure phase $e^{i\theta}$.