The Cartesian coordinate system and graph

To locate a point in a plane, you need two numbers: where the point is located horizontally, and where the point is located vertically. See Fig. 1. We have two axes: the x-axis, which is horizontal, and the y-axis, which is vertical. For each axis, the numbers are labeled. The numbers labeled in x-axis increase as you go rightward and decrease as you go leftward. The numbers labeled in y-axis increase as you go upward and decrease as you go downward. The x-axis meets the y-axis at O, which is called the "origin," and is where both of the labeled numbers on two axes are zero. You can locate a point by the two numbers on the axes. For example, see P_1. Its x-coordinate is 2, since you meet at 2 on the x-axis if you vertically travel from P_1 to the x-axis. Also, its y-coordinate is 3, since you meet at 3 on the y-axis if you horizontally travel from P_1 to the y-axis. Altogether, we say the coordinates of P_1 is $(2, 3)$. Also, there is no reason why the coordinates have to be positive and integers. For example, P_2's coordinate is $(-3, 1.5)$. There are other examples: P_3's coordinate is $(-2.4, 2)$ and P_4's coordinate is $(1.7, -2.6)$. Given that we now know what the Cartesian coordinate system is, let's use it to draw graphs. For example, let's draw $y = x$. This means that the y coordinate is equal to the x coordinate. See Fig. 2. The graph is a collection of such points. P_5 whose coordinates are $(1, 1)$ and P_6 whose coordinates are $(-3, -3)$ are two examples of such points. Notice also that the line passes through the origin as $y = 0$ when $x = 0$.

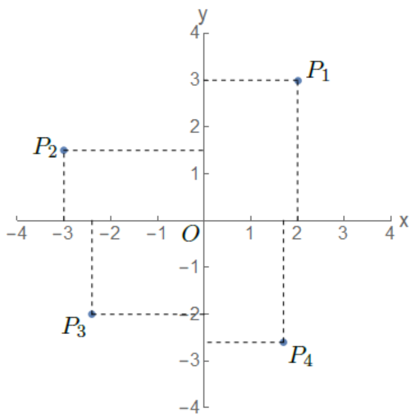

Figure 1: Cartesian coordinate system

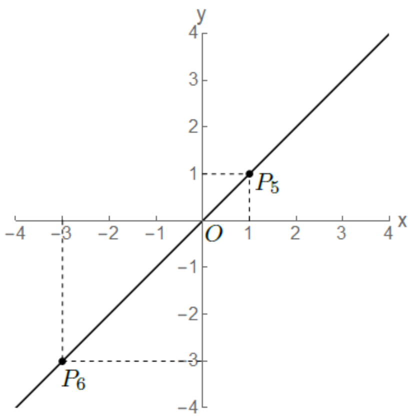

Figure 2: $y = x$

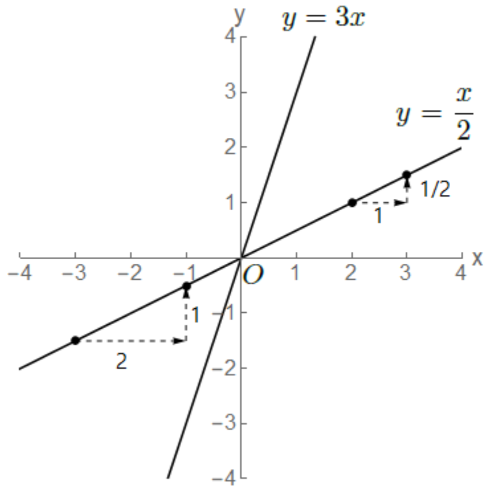

Figure 3: $y = 3x,\ \frac{x}{2}$

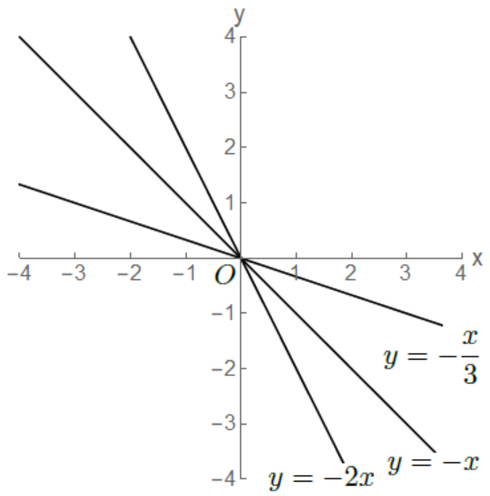

Figure 4: $y = -2x,\ -x,\ -\frac{x}{3}$

In Fig. 3. you see other graphs. We have $y = 3x$, $y = \frac{x}{2}$. One can easily see that the graph $y = 3x$ is very steep while the graph $y = \frac{x}{2}$ is not steep. Mathematically, the slope is defined by the ratio of the y change (Δy, pronounced "delta y") to the x change (Δx, pronounced "delta x").

$$\text{slope} = \frac{\Delta y}{\Delta x} \tag{1}$$

For example, let's calculate the slope of $y = 3x$. When x is equal to 0, y is also equal to 0. When x is equal to 2, y is equal to 6. While x increases by 2, y increases by 6. Thus, the slope is

$$\text{slope} = \frac{6-0}{2-0} = 3 \tag{2}$$

Now, let's find the slope of the graph $y = \frac{x}{2}$. In the figure, you see that as x changes by 1 from 2 to 3, y changes by 1/2 from 1 to 3/2. Thus, the slope is

$$\text{slope} = \frac{3/2-1}{3-2} = \frac{1/2}{1} = \frac{1}{2} \tag{3}$$

Let's find the slope of the same graph in another region. In the figure, you see that as x changes by 2 from -3 to -1, y changes by 1 from $-3/2$ to $-1/2$. Thus, the slope is

$$\text{slope} = \frac{-1/2-(-3/2)}{-1-(-3)} = \frac{1}{2} \tag{4}$$

Remarkably, the slope is the same. Actually, it is not hard to show that the slope of this graph is always 1/2, no matter which points you choose. When $x = x_1$, $y = x_1/2$. When $x = x_2$, $y = x_2/2$. Thus, the slope is

$$\text{slope} = \frac{x_2/2 - x_1/2}{x_2 - x_1} = \frac{(x_2 - x_1)/2}{x_2 - x_1} = \frac{1}{2} \tag{5}$$

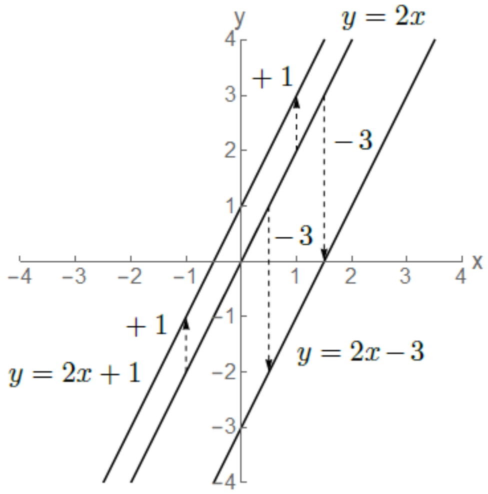

Figure 5: $y = 2x + 1,\ 2x,\ 2x - 3$

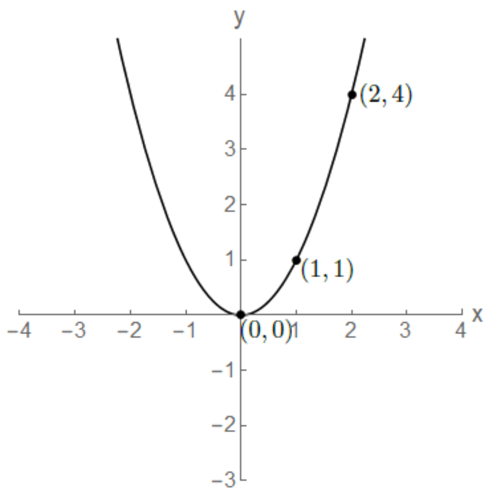

Figure 6: $y = x^2$

What is the consequence of the fact that this graph has a constant slope? It implies that this graph is a straight line, as we now prove.

Notice that we had two "right triangles" in our calculation. (3) is a right triangle with base 1 and height 1/2. (4) is a right triangle with base 2 and height 1. These two right triangles are necessary similar, because the ratio of two sides (i.e., the slope) are the same, and the angle between them are the same (i.e., the right angle). (SAS similarity). Thus, each corresponding angle of two right triangles are the same, which means that the graph doesn't change its direction and is a straight line.

More generally, the slope of the graph for $y = mx$ is given by m. This can be easily seen by generalizing (5) as follows. As the graph passes (x_1, mx_1) and (x_2, mx_2), we have

$$\text{slope} = \frac{mx_2 - mx_1}{x_2 - x_1} = m \tag{6}$$

By the way, there is no reason why the slope has to be always positive. For example, see Fig. 4. We have $y = -2x$, $y = -x$ and $y = -\frac{x}{3}$. The slopes are, respectively, -2, -1 and $-\frac{1}{3}$.

Notice that so far we have considered the graph of type $y = mx$. In other words, y divided by x (i.e. the ratio of y to x) is m. When y divided by x is constant, we say y is "proportional" to x. In such a case, if you double the x coordinate, the y coordinate is doubled. The ratio of y to x remains the same. Similarly, if you triple x, y is tripled. The ratio remains the same. If you halve x, y is halved. The ratio remains the same. Also, if x is 0, y is also 0 as m times 0 is always 0. Given this, let's consider a slightly different type of graph. See Fig.5. We have a graph $y = 2x$ the type we have already considered, and another type of graph: $y = 2x+1$ and $y = 2x-3$. We see that $y = 2x+1$ is obtained by moving $y = 2x$ by 1 upward. Similarly, we see that $y = 2x-3$ is obtained by moving $y = 2x$ by 3 downward.

Now, is y proportional to x in this other type of graph? The answer is no. Let's check $y = 2x + 1$ first. When $x = 1$ we have $y = 3$. When $x = 2$ we have $y = 5$. While x is doubled, y is not doubled. Therefore, y is not proportional to x. Similarly, you can also check that y is not proportional to x if $y = 2x - 3$. The y coordinate is proportional to the x coordinate only in the case in which $y = mx$ is satisfied.

How about the slope of these graphs? Let's check $y = 2x + 1$ first. When x increased by 1 from 1 to 2, y increase by 2 from 3 to 5. Therefore, the slope is 2 as 2 divided by 1 is 2. Similarly, one can check that the slope of $y = 2x - 3$ is also 2. More generally, the slope of the graph of the form $y = mx + b$ is m. Let's check it. When $x = x_1$ we have the y coordinate is $mx_1 + b$. When $x = x_2$ the y coordinate is $mx_2 + b$, thus the slope is

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{mx_2 + b - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = m \tag{7}$$

Thus, we see that b cancels, and the slope doesn't depend on b, and is simply given by m.

Actually, we have already witnessed this. Remember Fig. 5. We added 1 to or subtracted 3 from $y = 2x$, which resulted in the parallel movement of the graph.

Let me add a comment, at this point. We see that the slope, the ratio of y change to x change is constant (i.e. independent of Δx) for the graph of the form $y = mx + b$. However, for all the other types of graph, such as $y = x^2$, the slope is not constant and depends on where the points lie in the graph. See Fig. 6. In this case, we have $y = 0$ for $x = 0$, $y = 1$ for $x = 1$. Therefore y increases by 1 while x increases by 1. So, the slope seems to be 1. On the other hand, we have $y = 4$ when $x = 2$. Therefore when y increases by 3 from $y = 1$ to $y = 4$ while x increases by 1 from $x = 1$ to $x = 2$. So, the slope seems to be 3. In other words, we see that the slope is not constant. Indeed, if you look at the graph carefully, you see that the line gets steeper and steeper for bigger x. How the slope can be calculated in cases such as this is what you learn in the first day of class in "calculus," a very important subject in mathematics, which finds wide applications in science and engineering. We promise to talk about the slope in such cases when we teach you calculus in later articles.

We can also obtain a solution to an equation using the graph. For example, let's solve $2x + 3 = 3x + 4$. See Fig. 7. We can solve this equation by drawing two graphs: $y = 2x + 3$ and $y = 3x + 4$. At all the points on the graph $y = 2x + 3$, their x and y coordinates satisfy $y = 2x + 3$. At all the points on the graph $y = 3x + 4$, their x and y coordinates satisfy $y = 3x + 4$. Thus, at the point where these two graphs meet both $y = 2x + 3$ and $y = 3x + 4$ are satisfied. Thus, at this point, we have

$$y = 2x + 3 = 3x + 4 \tag{8}$$

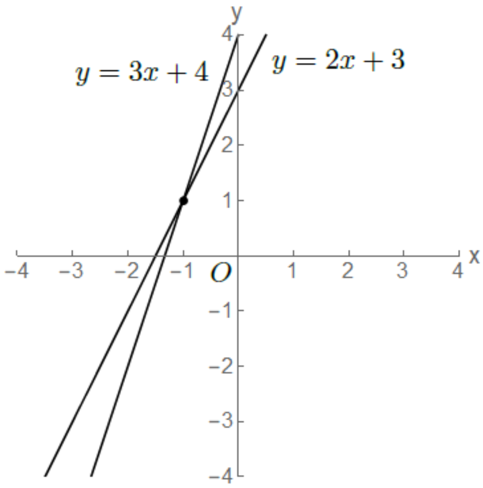

Figure 7: $y = 2x + 3,\ y = 3x + 4$

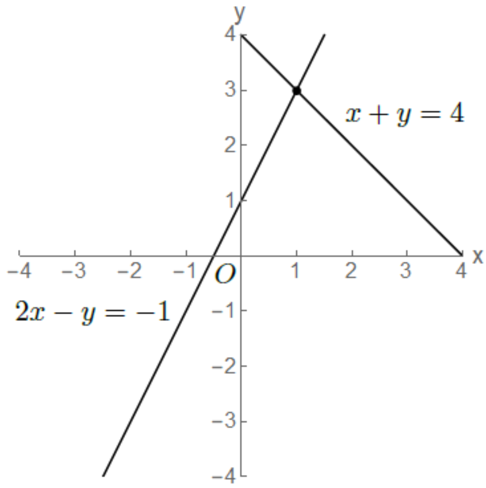

Figure 8: $2x - y = -1,\ x + y = 4$

From the graph we see that they meet at $(-1, 1)$ (i.e. when $x = -1$ and $y = 1$.) So, the answer is $x = -1$.

Also, there are other ways to express the same graph slightly differently. For example, $y = 2x + 1$ is the same graph as $2x - y = -1$. We can see this as:

$$y = 2x + 1 \tag{9}$$

$$y - 1 - y = 2x + 1 - 1 - y \tag{10}$$

$$-1 = 2x - y \tag{11}$$

This suggests that we can solve equations with two unknowns using graph. For example, we can solve $2x - y = -1$, and $x + y = 4$ by drawing the graphs for both equations and finding the intersection. See Fig. 8. They intersect at (1,3). Therefore, the answer is $x = 1$ and $y = 3$. Of course, in practice, this is not a good method to solve equations, but this is a good mental picture to have to visualize solving equations.

So far, we had two axes: x-axis and y-axis. However, it is possible to add one more axis, the z-axis. Then we can actually describe a three dimensional space, instead of a plane as we did before. For example, we can locate a point in space by three numbers. See Fig. 9. We see a point Q which has (1,2,4) as the coordinate.

Final comment. The 17th century French philosopher René Descartes was known for often staying in his bed until noon. One day, while staying in his bed, he found a fly crawling on the ceiling. Then, he suddenly figured out how to tell someone else where the fly was. He found out that he could tell them how far the fly was from each wall. That's how he came up with the Cartesian coordinate, named after him.

Problem 1. Check that the following graph has a constant slope m, and passes

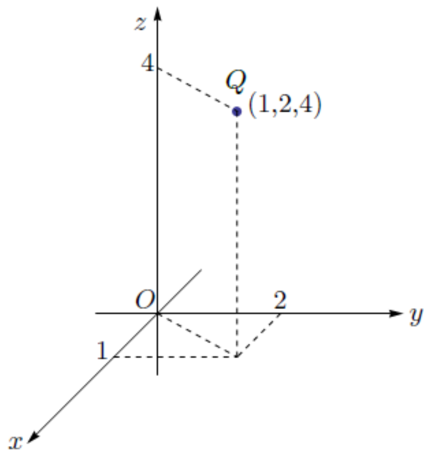

Figure 9: 3-dimensional Cartesian coordinate system

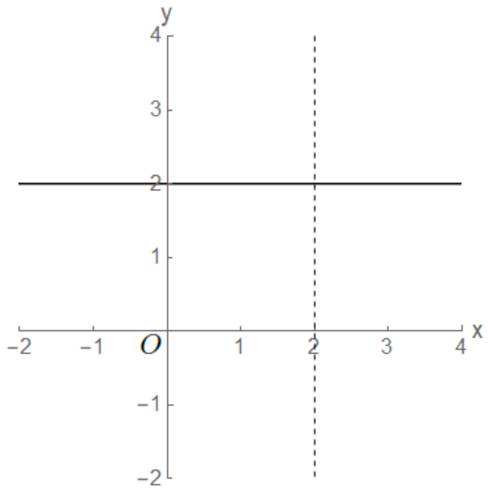

Figure 10: $x = 2,\ y = 2$

(x_0, y_0).

$$y - y_0 = m(x - x_0) \tag{12}$$

Problem 2. Let's say that a graph of the form $y = mx + b$ passes (x_0, y_0). Express b in terms of x_0, y_0, and m. (Hint: You can use the result of Problem 1.)

Problem 3. A graph passes (2,0) and has a constant slope of -2. What is the equation of this graph? Express it in the form $y = mx + b$. (Hint: You can use the result of Problem 1 or 2.)

Problem 4. Let's say that a graph of the form $y = mx + b$ passes these two points:

$$(1, 2), \quad (7, 6) \tag{13}$$

What are m and b? (Hint: First, you can find its slope by considering how much y increases when x increases from 1 to 7. Then, you can use the result of Problem 1 or Problem 2.)

Problem 5. Let's say that a graph of the form $y = mx + b$ passes these two points:

$$(3, 2), \quad (7, -2) \tag{14}$$

What are m and b?

Problem 6. In Fig. 10 we have two graphs: one with a dotted line, and one with a solid line. One of them is $x = 2$ and the other $y = 2$. Which one is which? Can you also draw $x = 0$ and $y = 0$ on the graph? (The answer to the last question can be found in the summary.)

Problem 7. In Fig. 11 we have three graphs: A, B, C. They are $y = x^2, y = 2x^2, y = \frac{x^2}{2}$. Which one is which? Notice that y can be never negative for all three

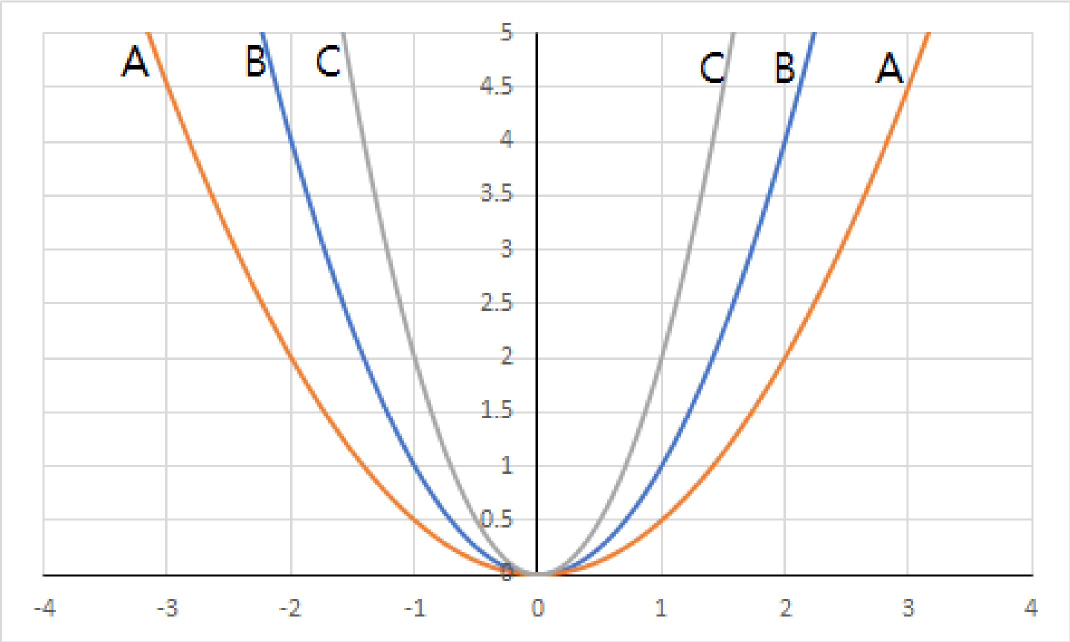

Figure 11: $y = x^2, 2x^2, \dfrac{x^2}{2}$

graphs as something squared multiplied by a positive number is always non-negative, and can be 0, which is the minimum only when x is 0.

Here, you see that C is sharper than B, and B is sharper than A. If you correctly solve this problem, you will find that, for positive a and b, if a is bigger than b, $y = ax^2$ is sharper than $y = bx^2$.

Summary

- In Cartesian coordinate, x-axis is horizontal, while y-axis vertical.
- The x-axis meets the y-axis at O, which is called the "origin," and is where both of the labeled numbers on two axes are zero.
- The slope is defined by the ratio of the y increase to the x increase.
- The slope of the graph of form $y = mx$ is m.
- When y divided by x is constant, we say y is "proportional" to x.
- The slope of the graph of the for $y = mx + b$ is m.
- We can describe a three dimensional space by using three dimensional Cartesian coordinate; in addition x-axis and y-axis, we add z axis.
- In 2-dimensional Cartesian coordinate, $y = 0$ corresponds to the x-axis, and $x = 0$ corresponds to the y-axis.