The Pythagorean theorem

See Fig. 1. Here, we have a right triangle. The length of the hypotenuse (the side opposite the right angle) is given by c and the length of the other two sides are a and b. What is the relation between a, b and c? How can you obtain the length of the hypotenuse c, given the other two sides a and b? The Pythagorean theorem (also known as Pythagoras's theorem) answers this question. Pythagoras was a Greek philosopher in the 6th century B.C. To derive the Pythagorean theorem, see Fig.2. Here, we have a big right square with side $(a+b)$. Inside this big square, there is a small quadrangle with side c. What are the angles of the quadrangle? First, from Fig.1 notice that $A+B+90° = 180°$. Also, from Fig. 2 we have $A+B+D = 180°$. Therefore, we conclude $D = 90°$. This proves that the quadrangle is actually a square. Therefore, its area is given by c^2. On the other hand, what is the area of the square in terms of a and b? The area of the big square is $(a+b)^2$. The area of each triangle is $\frac{1}{2}ab$. As the area of small square is the area of big square subtracted by the area of four triangles, we have:

$$\begin{aligned} c^2 &= (a+b)^2 - 4 \times \frac{1}{2}ab && (1) \\ &= a^2 + 2ab + b^2 - 2ab = a^2 + b^2 && (2) \end{aligned}$$

Therefore, we conclude $a^2 + b^2 = c^2$. This is the Pythagorean theorem.

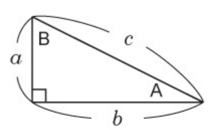


Fig. 1

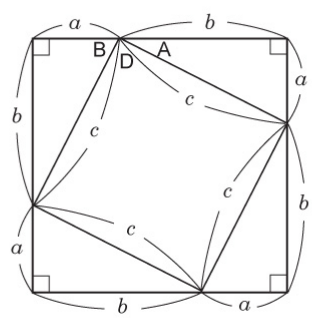


Fig. 2

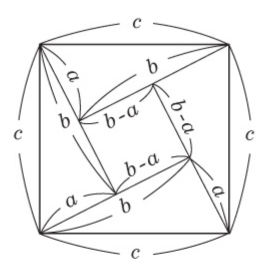


Fig. 3

Actually, there are dozens ways to derive the Pythagorean theorem. I present another one in Fig. 3. Here, we have a big quadrangle with side c and inside it we have a small square with side $b-a$. The big quadrangle is actually a square as the sum of A and B in Fig.1 is $90°$. Therefore, the area of the big square is c^2. What is this in terms of a and b? The area of the small square is $(a-b)^2$ and the area of each triangle is $\frac{1}{2}ab$. As the area of the big square is the sum of the area of the small square and four triangles, we have:

$$c^2 = (a-b)^2 + 4 \times \frac{1}{2}ab \qquad (3)$$

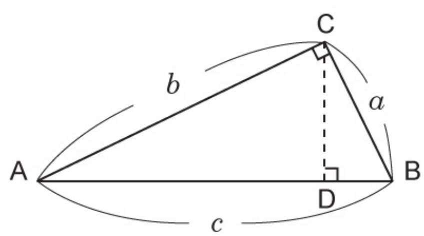
$$= \quad a^2 - 2ab + b^2 + 2ab = a^2 + b^2 \tag{4}$$

Final comment. If a and b are two sides of triangle, and the angle between them is $90°$, we know that the other side c is given by $\sqrt{a^2+b^2}$. That is what we have learned in this article. However, what if the angle between them is not $90°$ but another angle θ? We will find the answer to the question in our later article "The law of cosines."

Problem 1. In the following list, each set of three numbers denotes the three sides of a triangle. Determine which are right triangles, and which aren't.

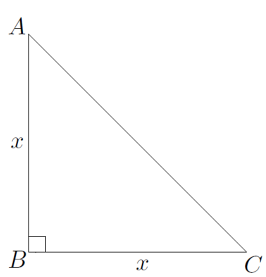
$$(1, 1, \sqrt{2}) \quad (3, 4, 5) \quad (4, 5, 7) \quad (5, 5, 7) \quad (5, 12, 13) \quad (6, 8, 10) \quad (\sqrt{3}, \sqrt{3}, 3) \quad (4, 6, 8)$$

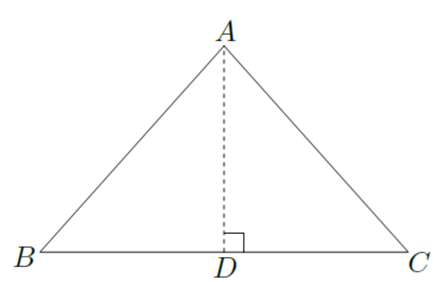
Problem 2. This problem asks you to prove the Pythagorean theorem in an alternative way. See the figure below. First, show that ΔABC and ΔACD are similar. Also, show that ΔABC and ΔCBD are similar. Using these similarities, obtain the length of $\overline{AD}$ and $\overline{DB}$ in terms of a, b and c. Then, obtain the relation $a^2 + b^2 = c^2$ from the condition that the sum of the length of $\overline{AD}$ and the length of $\overline{DB}$ must be equal to c.



Problem 3. Consider $\triangle ABC$. We have $\overline{AB} = 6$, $\overline{BC} = 8$, and $\overline{AC} = 11$. Is $\angle B$ bigger than the right angle or smaller? (Hint[1]) Consider $\triangle DEF$. We have $\overline{DE} = 7$, $\overline{EF} = 8$, $\overline{DF} = 11$. Is $\angle E$ the right angle or bigger than the right angle or smaller? Now, consider the list in Problem 1. Among the triangles that are not right triangles, can you tell which ones are acute triangles and which ones are obtuse triangles? Actually, we will later learn how to determine all the angles of triangles once we know all the sides of triangles.

Problem 4. See the right isosceles triangle ABC in the left figure below. If the length of $\overline{AB}$ is x, then we know that the length of $\overline{BC}$ is also x. Then, what is the length of $\overline{AC}$?





[1] If $\overline{AC}$ were 10, it would have been a right triangle.

Problem 5. See the isosceles triangle in the right figure above. Both the length of $\overline{AB}$ and the length of $\overline{AC}$ are 3, and the length of $\overline{BC}$ is 4. Then, what is the area of $\triangle ABC$? (Hint[2])

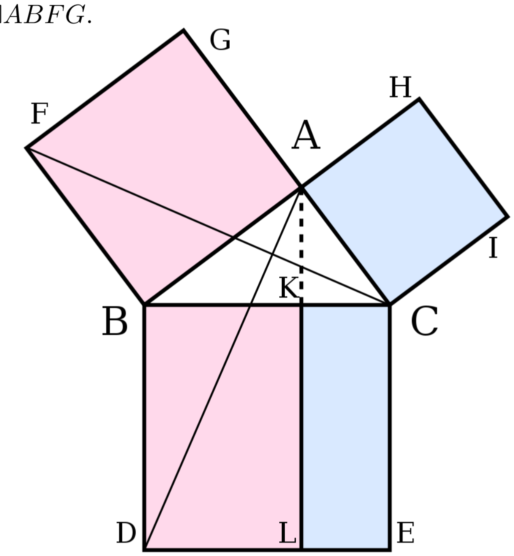
Problem 6. In this problem, we introduce yet another proof of the Pythagorean theorem. This one is from "the Elements" the very famous mathematics textbook written by the ancient Greek mathematician Euclid. See the figure below. $\triangle ABC$ is a right triangle. $\square ABFG$, $\square ACIH$, and $\square BCED$ are all right squares. Thus, if we show that the area of $\square BCED$ is the sum of the area of $\square ABFG$ and the area of $\square ACIH$ then the Pythagorean theorem will be proven. To prove this, you will be required to show that the area of $\square ABFG$ is equal to the area of $\square BKLD$ (the red ones in the figure). Here, $\overline{AK}$ is perpendicular to $\overline{BC}$. Then, one this is shown, by the same token, the area of $\square ACIH$ is equal to $\square CKLE$, which will complete the proof (the blue ones in the figure). So, let's show this.

(a) Explain why the area of $\triangle BCF$ is half the one of $\square ABFG$.

(b) Explain why $\triangle BCF$ is congruent to $\triangle BDA$.

(c) Explain why the area of $\triangle BDA$ is half the one of $\square BKLD$.

(d) From (a), (b), and (c) conclude that the area of $\square BKLD$ is equal to the one of $\square ABFG$.

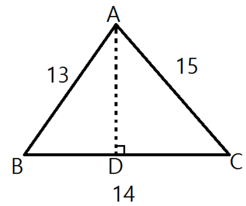


Problem 7. See the figure below. The triangle ABC has three sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, with lengths 13, 14, and 15 respectively. Find the area of the triangle by calculating the height $\overline{AD}$. (Hint[3]) Notice that you can apply the method to solve this problem to any other

[2]First, find the height $\overline{AD}$, by noticing that the lengths of $\overline{BD}$ and $\overline{DC}$ are same, because $\triangle ABC$ is an isosceles triangle.

[3]Let's denote the length of the line segment $\overline{BD}$ by x. Then, you will be able to express the height in terms of x by applying the Pythagorean theorem to $\triangle ABD$. Also, you will be able to express the height another way by applying the Pythagorean theorem to the triangle ADC. This time, you will first need to express the length of line segment $\overline{DC}$ in terms of x. As you have two expressions for height in terms of x you will be able to solve x by equating these two expressions. Once, you find x, you can plug the value into these two expressions to obtain the height. Then, the length of $\overline{BC}$ multiplied by the half of height will be

triangles when the lengths of each side are known.



Summary

- Consider a right triangle. If the length of the hypotenuse is given by c and the length of the other two sides are a and b, we have $a^2 + b^2 = c^2$. This is called the Pythagorean theorem.

the area.