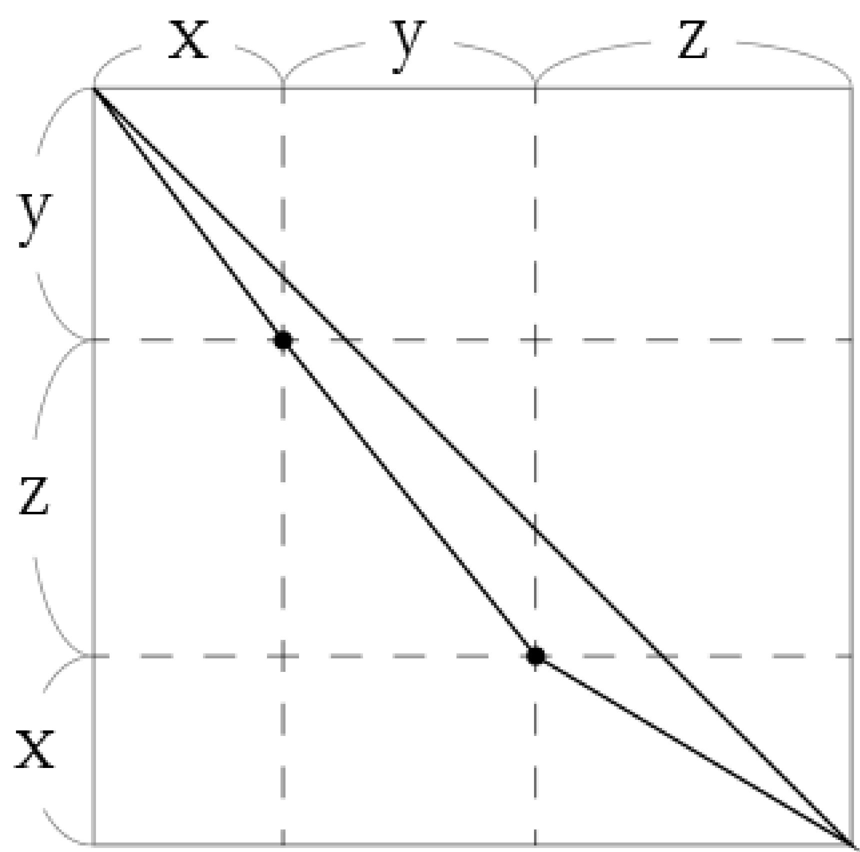
Geometric proofs for algebraic problems

In earlier articles, we have seen how we can use algebraic methods to solve geometric problems. In this article, we will do the exact opposite. We will use geometric methods to solve two algebraic problems. The first problem is proving the following:

$$\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2} \geq \sqrt{2}(x+y+z) \tag{1}$$

Here, I present a proof that I came across on Youtube.[1] The point of this article is to emphasize again that there is no "mysterious mental faculty" that can be used to crack a mathematics problem all at once, as we mentioned in our essay "Is math and science homework mechanical drudgery?" One always has to come up with clever tricks.

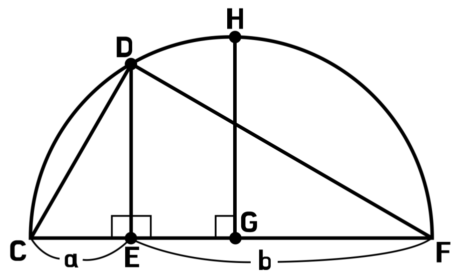
The proof is self-explanatory. See the figure.



[1] *Easy proof for a hard inequality* by *Ultra.p Maths.* https://www.youtube.com/watch?v=hdNCCUkRRms.

Problem 1. When does the equality hold in (1)? (Hint: the figure can be very helpful.)

The second problem is proving that the arithmetic mean is bigger than or equal to the geometric mean. See the figure below for a triangle inscribed in a half circle. G is the center.



Problem 2. What is the radius of this half circle? What is the length of $\overline{GH}$?

Problem 3. If you solved Problem 2 of our earlier article "Inscribed quadrilateral in a circle" correctly, you will see that $\angle CDF$ is a right angle. Explain why $\triangle CDE$ is similar to $\triangle DFE$.

Problem 4. Using this fact, obtain the length of $\overline{DE}$.

Problem 5. From the fact that $\overline{GH}$ is longer than or equal to $\overline{ED}$, conclude that the arithmetic mean is always greater than or equal to the geometric mean.

Summary

- Algebraic problems can be sometimes solved by geometric methods.