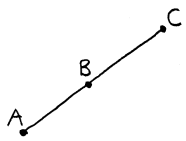
An algebraic proof of the triangle inequality

The triangle inequality states that the length of any one side of a triangle is always smaller than the sum of the lengths of the other two sides. I assume that you are familiar with this inequality from our earlier article "The triangle inequality." If not, don't worry. I will prove it for you in this article.

Suppose you pick any three points A, B and C on a plane and connect these points to form a triangle. If a triangle is indeed formed, the triangle inequality will be satisfied (as we showed in our earlier article and will show again shortly).

However, there are cases in which a triangle is not formed, even though you connect these points. This is when the three points lie on a straight line. See the figure.



If you count this object as a "triangle" even though you are not technically allowed to do so, as it is a fake one, you see that its longest "side" (i.e. $\overline{AC}$) is equal to the sum of the other two "sides" (i.e. $\overline{AB} + \overline{BC}$). If you include this case, we can say, for *any* points A, B and C (i.e., not just when A, B and C are *not* on a straight line, and thus form a *real* triangle) the distance between A and C is always less than or equal to the sum of the distance between A and B and the distance between B and C. Notice that the phrase "equal to" is included in the above statement to account for the case depicted in the figure. In any case, if we denote the coordinate of A as (A_x, A_y) and similarly for B and C, this statement can be translated into formulas as follows:

$$\sqrt{(C_x - A_x)^2 + (C_y - A_y)^2} \le \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2} + \sqrt{(C_x - B_x)^2 + (C_y - B_y)^2}$$

We can actually prove the above inequality by using the Cauchy-Schwarz inequality. First, let's set $B_x - A_x = X_1$, $B_y - A_y = Y_1$, $C_x - B_x = X_2$, and $C_y - B_y = Y_2$. Then, the above inequality can be re-written as:

$$\sqrt{(X_1 + X_2)^2 + (Y_1 + Y_2)^2} \le \sqrt{X_1^2 + Y_1^2} + \sqrt{X_2^2 + Y_2^2}$$

Squaring both sides, and subtracting common terms we get: (**Problem 1.** Check this!)

$$X_1X_2 + Y_1Y_2 \leq \sqrt{X_1^2 + Y_1^2}\sqrt{X_2^2 + Y_2^2} \tag{1}$$

However, in our article on the Cauchy-Schwarz inequality, we proved the following:

$$(X_1X_2 + Y_1Y_2)^2 \leq (X_1^2 + Y_1^2)(X_2^2 + Y_2^2) \tag{2}$$

Taking the square root of both sides, we conclude (1) is correct. Thus, we have proved the triangle inequality.

Problem 1. We will consider now the three-dimensional version of our earlier problem in this article. Let's say that the coordinate of A is (A_x, A_y, A_z), the coordinate of B is (B_x, B_y, B_z) and the coordinate of C is (C_x, C_y, C_z). Using the following Cauchy-Schwarz inequality for three variables

$$(X_1X_2 + Y_1Y_2 + Z_1Z_2)^2 \leq (X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2), \tag{3}$$

prove that the distance between A and C is less than equal to the sum of the distance between A and B and the distance between B and C.

Summary

- The triangle inequality can be proven by using the Cauchy-Schwarz inequality.