Area of a circle

Those of you who graduated from middle school will remember that the area of a circle is given by πr^2, where r is the radius of the circle. But, do you remember why? Maybe, you remember it as follows. See Fig. 1. There is a big square with area $4r^2$ and four smaller squares with area r^2. The circle does not cover the whole big square, so its area is slightly less than $4r^2$. Maybe, something around $3r^2$, or more precisely, πr^2.

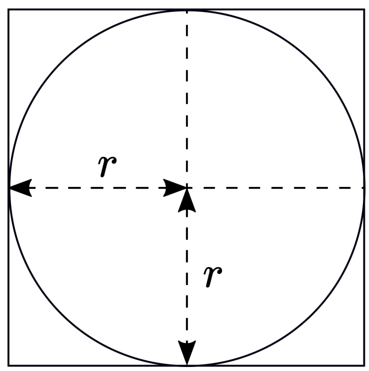

Figure 1: Square with an area of $4r^2$ with a circle of radius r inscribed.

You are not certainly wrong, but your argument is not entirely correct either. Recall that π is defined by the circumference-to-diameter ratio not by the area of a circle-to-r^2 ratio. Nevertheless, the area of a circle divided by r^2 is also π. In other words, you have to justify that the value appearing in the area of a circle is exactly the circumference-to-diameter ratio.

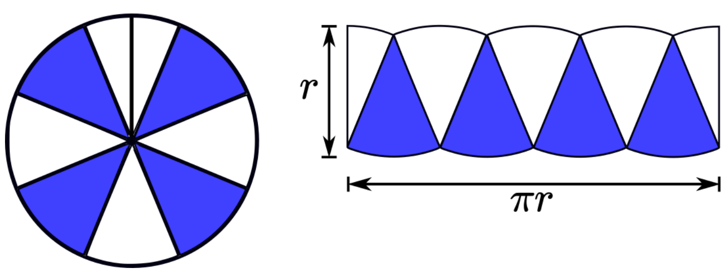

Figure 2: Circle divided into 8 equal sectors (left). Circle's sectors aligned, forming a rectangle-like shape.

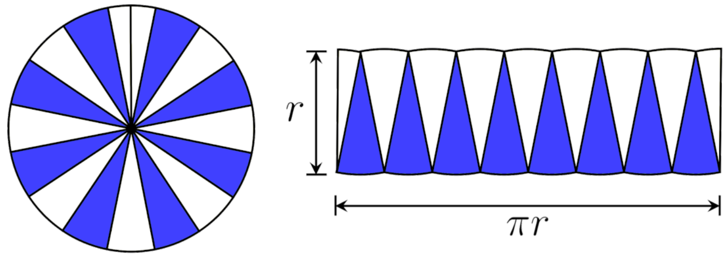

Figure 3: The same circle as in Fig. 2, but divided into 16 equal sectors.

So, let me remind you why the area of a circle is πr^2. See Fig. 2. We divided a circle into eight equal sectors. Four of them are blue and four of them are white. One of the white sectors is further divided into two equal smaller sectors. Then, we rearrange these sectors as on the right. It looks close to a rectangle with a length of half of the circumference ($\pi r = 2\pi r/2$) and a width of its radius. Maybe, not. Or, quite? See Fig. 3. If we divide the circle into 16 equal sectors instead of 8, the rearranged figure looks much more like a rectangle than before. Actually, it is not hard to imagine that the resulting figure looks more and more like a rectangle as we divide the circle into smaller and smaller sectors and rearrange them.

As the "rectangle" has the length πr and radius r, the area of the circle is

$$A = \pi r \times r = \pi r^2 \tag{1}$$

Let me present another derivation of the area of a circle. This one is due to Archimedes. See Fig. 4. An octagon is inscribed in a circle. The triangle marked in the figure has an area

$$A_{\text{triangle}} = \frac{1}{2} sh \tag{2}$$

The octagon has 8 such triangles. Thus, its area is given by

$$A_{\text{octagon}} = 8 \times A_{\text{triangle}} = \frac{1}{2}(8s)h = \frac{1}{2}Ch \tag{3}$$

Here, $C = 8s$ is the perimeter of the octagon. It is slightly smaller than the circumference of the circle, $2\pi r$. Similarly, h is slightly smaller than the radius of the circle. It is not hard to imagine that, as you increase the sides of the regular polygon, its perimeter will approach to the circumference of the circle, and h will approach its radius. You will know what I mean, if you read our essay "The calculation of π, the first part." Thus, we have

$$A_{\text{circle}} = \frac{1}{2}(2\pi r)r = \pi r^2 \tag{4}$$

In a similar way, see Fig. 5. You see a circle with an octagon circumscribed. If the side of the octagon is s', the area of the triangle marked in the figure is $(1/2)s'r$. As before, the total area of the octagon is

$$A'_{\text{octagon}} = 8 \times A'_{\text{triangle}} = \frac{1}{2}(8s')r = \frac{1}{2}C'r \tag{5}$$

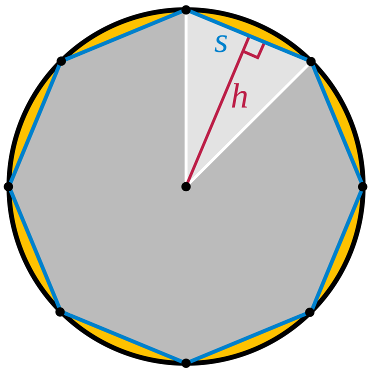

Figure 4: A circle with an octagon inscribed [1].

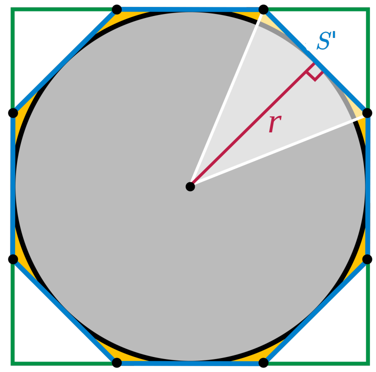

Figure 5: The same circle as in Fig. 4 but now with the octagon circumscribed [2].

As we increase the number of sides of the regular polygon, its perimeter will get closer and closer to the circumference of the circle. Thus, we come to the formula (4) again and we obtain πr^2 once again.

When referring to Archimedes's method, why did I consider the two separate cases? Inscribed polygon and circumscribed polygon? In this article, we were quite sloppy about our arguments, and just said that the area of the regular polygon approaches the area of the circle as we consider the regular polygons with more sides. However, in the original Archimdes's method, he was rigorous. From the inscribed polygons (i.e., Fig. 4), he rigorously proved that the area of a circle is not greater than πr^2, and from the circumscribed polygons (i.e., Fig. 5), he rigorously proved that the area of a circle is not less than πr^2. The only possiblity we can have is that the area of a circle is equal to πr^2.

Summary

- The area of a circle with radius r is given by πr^2.

References

[1] Adapted from `https://en.wikipedia.org/wiki/File:Archimedes_circle_area_proof_-_inscribed_polygons.svg`

[2] Adapted from `https://en.wikipedia.org/wiki/File:Archimedes_circle_area_proof_-_circumscribed_polygons.svg`