Arithmetic series: $1 + 2 + 3 + \cdots + 99 + 100 =?$

The legend has it that Gauss, one of the most famous mathematicians of all time, was asked to sum up the numbers from 1 to 100 by his teacher when he was nine years old. When he obtained the answer very quickly, he astonished his teacher. Here, we explain his trick.

We want to calculate the following:

$$S = 1 + 2 + 3 + \cdots + 98 + 99 + 100 \tag{1}$$

Now, let's re-write this slightly differently. We have:

$$S = 100 + 99 + 98 + \cdots + 3 + 2 + 1 \tag{2}$$

If we add them up side by side,

$$S+S = (1+100)+(2+99)+(3+98)+\cdots+(98+3)+(99+2)+(100+1) \tag{3}$$

which implies

$$2S = 101 + 101 + 101 + \cdots + 101 + 101 + 101 \tag{4}$$

Since S has 100 terms, the above formula must also have a hundred 101s. Therefore,

$$2S = 101 \times 100 \tag{5}$$

We conclude $S = 5050$. So, we see that we do not need to sum everything one by one to obtain the answer.

Now, we introduce a mathematical notation. It would be cumbersome to write out a sum of sequence such as S in (1). So, we have a short hand notation:

$$\sum_{k=1}^{n} a_k \equiv a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \tag{6}$$

where $\equiv$ means "defined by," i.e the left-hand side is defined by the right-hand side. Here, we see that k takes values from 1 to n, and they are all added together.

Using this notation, we can re-express (1) as:

$$S = \sum_{k=1}^{100} k \tag{7}$$

The trick introduced in this article can be only applied to "arithmetic series." A "series" is a sum of a sequence, and an arithmetic series is a sum of an arithmetic sequence. An arithmetic sequence is a sequence that has a constant difference between consecutive terms. For example,

$$1, 4, 7, 10, 13, \cdots \tag{8}$$

is an arithmetic sequence.

In South Korea, most 10th graders also learn how to calculate formulas such as:

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3 \tag{9}$$

and, they are required to memorize the answer to them. It's nice to learn how to calculate them, but I don't think that they should be required to memorize them, especially considering that they aren't provided with formula booklets in college entrance exams.

In the next article, we will consider geometric series. A geometric series has a constant ratio between successive terms.

Problem 1. Obtain the following T. (Hint[1])

$$T = \sum_{k=1}^{20} (2k-1) = ? \tag{10}$$

Summary

- $$\sum_{k=1}^{n} a_k \equiv a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$
- A "series" is a sum of a sequence.
- An arithmetic sequence is a sequence that has a constant difference between consecutive terms.
- To calculate an arithmetic series, you can first write out the arithmetic sequence in the original order in the first line, and write out the arithmetic sequence in the reverse order in the second line. Then, you can add side by side, which gives out constant terms. Finally, you can get the answer by summing up these constant terms by a simple multiplication and divide by two.

[1] Use $T = 1 + 3 + \cdots + 39, \quad T = 39 + 37 + \cdots + 1$