# Arithmetic mean, geometric mean and harmonic mean

Let's say that my daily salary two years ago was 200 dollars. This year, it is 288 dollars as it has risen 44 % last 2 years. On average, at what percent did my salary rise per year? At first glance, it seems 22% as 44 divided by 2 is 22. However, this is not the correct answer. If salary rose 22% per year for the last two years, the salary this year would have been

$$200 \times 1.22^2 = 297.68 \tag{1}$$

which is not 288. Let's find the correct answer. If the salary rose by $a$ times per year for two years from 200 dollars, the current salary would be $200a^2$. Thus, we need find $a$ that satisfies

$$200a^2 = 288 \tag{2}$$

As $a^2$ is 288/200=1.44. $a$ is 1.2. In other words, the salary has risen 20% per year. Then, what would be the salary last year, i.e., the year after when I received 200 dollars as my salary. It is 200×1.2=240 dollars. Notice that this is in a sense, a "mean" between my salary two years ago and my salary now, because the salary rose by fixed percent every year. We say 240 is a "geometric mean" of 200 and 288.

Suppose again my daily salary two years ago was 200 dollars and my salary today is 288 dollars. Let's say that my salary rose by fixed amount every year. Then, what would be my salary a year ago? As the salary rose 88 dollars for two years, it rose 44 dollars every year. Therefore, my salary last year was 200+44=244 dollars. Notice that this is the arithmetic mean between 200 and 288. We can calculate it by

$$200 + (288 - 200)/2 \tag{3}$$

or, more generally, if $a = 200$ and $b = 288$,

$$a + (b - a)/2 = a + b/2 - a/2 = a - a/2 + b/2 = a/2 + b/2 = (a + b)/2 \tag{4}$$

which is indeed the arithmetic mean of $a$ and $b$, and a formula you easily recognize.

For the geometric mean, we can calculate it by

$$200 \times \sqrt{\frac{288}{200}} \tag{5}$$

or more generally,

$$a \times \sqrt{\frac{b}{a}} = \sqrt{a^2}\sqrt{\frac{b}{a}} = \sqrt{ab} \tag{6}$$

Thus, the geometric mean of $a$ and $b$ is $\sqrt{ab}$.

**Problem 1.** Let's say your salary now is 100 dollars weekly, and your salary two years from now will be 2500 dollars weekly. If your salary is determined to increase by fixed amount what will be the salary next year? Or, if your salary is determined to increase by fixed percentage (i.e., fixed ratio) what will be the salary next year? Which one is bigger? arithmetic mean or geometric mean?

Actually, it turns out that the arithmetic mean is always bigger than the geometric mean. Let's prove this. We have to compare $(a+b)/2$ and $\sqrt{ab}$. Let's say that we do not know yet which one is bigger. So, let's write "?" instead of $\leq$ or $\geq$.

$$\frac{a+b}{2} \ ? \ \sqrt{ab} \tag{7}$$

which means

$$(a+b) \ ? \ 2\sqrt{ab} \tag{8}$$

In other words the sign in (7) is $\leq$, the sign in (8) is also $\leq$, and if the sign in (7) is $\geq$, the sign in (8) is also $\geq$. Then,

$$a - 2\sqrt{ab} + b \ ? \ 0 \tag{9}$$

$$(\sqrt{a} - \sqrt{b})^2 \ ? \ 0 \tag{10}$$

Something squared is always bigger than or equal to 0. Thus, we see that ? in (10) is $\geq$. Thus, the sign in (7) is also $\geq$. In other words, an arithmetic mean is always bigger than or equal to a geometric mean. Then, when is the arithmetic mean equal to the geometric mean? That is when ? in (10) is the equal sign. In other words, when $(\sqrt{a} - \sqrt{b})^2$ is zero, i.e., when $a = b$. We see that the arithmetic mean and the geometric mean of two numbers $a$ and $b$ are the same if and only if $a = b$.

**Problem 2.** For two positve numbers $a$ and $b$, prove similarly that the arithmetic mean is always bigger than or equal to the harmonic mean. When is the arithmetic mean equal to the harmonic mean?

**Problem 3.** For two positive numbers $a$ and $b$, prove similarly that the geometric mean is always bigger than or equal to the harmonic mean. When is the geometric mean equal to the harmonic mean?

In our later article "Geometric proofs for algebraic problems," we will present a geometric proof that the arithmetic mean is always bigger than or equal to the geometric mean. In that article, you will learn what a "geometric proof" is.

So far, we have only dealt with arithmetic mean, geometric mean and harmonic mean of two numbers. However, it can be generalized into arbitrary number of numbers. For example, the arithmetic mean, the geometric mean and the harmonic mean of 5 positive numbers $a$, $b$, $c$, $d$, $e$ is

$$\frac{a+b+c+d+e}{5}, \qquad \sqrt[5]{abcde}, \qquad \frac{5}{1/a+1/b+1/c+1/d+1/e} \tag{11}$$

Remarkably, even for arbitrary number of numbers, it is true that the arithmetic mean is always bigger than or equal to the geometric mean, which in turn is always bigger than or equal to the harmonic mean. As with the means of 2 numbers, the arithmetic mean is equal to the geometric mean, if and only if all the numbers are the same. The same can be said about the geometric mean and the harmonic mean.

We will now give the general proof for the inequality of the arithmetic mean and the geometric mean. For $n$ positive numbers $x_1$, $x_2$,$\cdots$,$x_n$, notice that the arithmetic mean and the geometric mean are equal if $x_1 = x_2 = \cdots = x_n$.

Given this, let's compare

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \qquad \text{and} \qquad \sqrt[n]{x_1 x_2 \cdots x_n} \tag{12}$$

for arbitrary positive numbers $x_1, \cdots, x_n$.

Now, among these numbers, choose the biggest number $x_i$ and the smallest number $x_j$. If $x_i = x_j$, all the numbers here are the same. Then, the arithmetic mean and the geometric mean are the same, as we mentioned, thus we proved that the arithmetic mean is greater than or equal to the geometric mean in this case.

So, let's consider the case $x_i > x_j$. Given such a pair, let's define "new" $x_i$ and $x_j$ by

$$x_i' = \frac{x_i + x_j}{2}, \qquad x_j' = \frac{x_i + x_j}{2} \tag{13}$$

In other words, $x_i'$ and $x_j'$, which are new $x_i$ and $x_j$, are the arithmetic mean of $x_i$ and $x_j$. Notice that $x_i' + x_j' = x_i + x_j$.

**Problem 4.** Show that the arithmetic mean doesn't change by this replacemnt. In other words, the arithmetic mean of $x_1, \cdots, x_i', \cdots x_j', \cdots, x_n$ is equal to $x_1, \cdots, x_i, \cdots x_j, \cdots, x_n$.

**Problem 5.** Show that $x_i' x_j'$ is bigger than $x_i x_j$.

In other words, while the arithmetic mean remains the same, the geometric mean increases.

After performing this step, we will have a new set of numbers, and we can choose the biggest and the smallest numbers and repeat this step again and again. Then, the arithmetic mean will always remain the same, while the geometric mean will always increase. After repeating this step sufficiently many times, the numbers will be "averaged" many times, which makes each number in the set close to one another. In other words, each number in the set is now very close to their arithmetic mean, which in turn is equal to the original arithmetic mean that we begun with. Also, their geometric mean, now, will be very close to their arithmetic mean (which is equal to the original arithmetic mean), as each number in the set is almost equal to one another.

To repeat what we just stated, if we denote the arithmetic mean now by $AM_{\text{now}}$, their original arithmetic mean by $AM_0$, their geometric mean now by $GM_{\text{now}}$, their original geometric mean by $GM_0$, we have

$$\text{each number in the set now} \approx AM_{\text{now}} = AM_0 \approx GM_{\text{now}} \tag{14}$$

Given this, remember that

$$GM_{\text{now}} > GM_0 \tag{15}$$

as performing each step increased the geometric mean.

Combining (14) and (15), we see that their original arithemtic mean is greater than their original geometric mean. In other words, we proved

$$\frac{x_1 + x_2 + \cdots + x_n}{n} > \sqrt[n]{x_1 x_2 \cdots x_n} \tag{16}$$

**Problem 6.** Consider $n$ positive numbers, $x_1, x_2, \cdots, x_n$. Prove that their geometric mean is bigger than or equal to their harmonic mean, by defining $y_i = 1/x_i$ and using the fact that the arithmetic mean of $y_1, y_2, \cdots, y_n$ is bigger than the geometric mean of the same numbers. (Hint[1])

That the arithemtic mean is always greater than or equal to the geometric mean comes handy when you solve some math olympiad type problems. Here is an example from Wikipedia.

For positive numbers $x$, $y$, $z$, what is the minimum of

$$f(x,y,z) = \frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}} \tag{17}$$

and for what values of $x$, $y$, $z$ is the minimum achieved?

The trick is re-expressing $f(x,y,z)$ as follows:

$$f(x,y,z) = 6 \cdot \frac{\frac{x}{y} + \frac{1}{2}\sqrt{\frac{y}{z}} + \frac{1}{2}\sqrt{\frac{y}{z}} + \frac{1}{3}\sqrt[3]{\frac{z}{x}} + \frac{1}{3}\sqrt[3]{\frac{z}{x}} + \frac{1}{3}\sqrt[3]{\frac{z}{x}}}{6} \tag{18}$$

$$= \frac{6\frac{x}{y} + 3\sqrt{\frac{y}{z}} + 3\sqrt{\frac{y}{z}} + 2\sqrt[3]{\frac{z}{x}} + 2\sqrt[3]{\frac{z}{x}} + 2\sqrt[3]{\frac{z}{x}}}{6} \tag{19}$$

$$= \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{6} \tag{20}$$

where

$$x_1 = 6\frac{x}{y}, \qquad x_2 = x_3 = 3\sqrt{\frac{y}{z}}, \qquad x_4 = x_5 = x_6 = 2\sqrt[3]{\frac{z}{x}} \tag{21}$$

Now, we can use

$$f(x) = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{6} \geq \sqrt[6]{x_1 x_2 x_3 x_4 x_5 x_6} \tag{22}$$

**Problem 7.** Complete the calculation, and obtain the minimum value for $f(x,y,z)$.

**Problem 8.** What are the values of $x$, $y$, $z$ when $f(x,y,z)$ is the minimum? (Hint[2])

**Problem 9.** Find the minimum for

$$g(x,y,z) = \sqrt{\frac{x}{y}} + \frac{2y}{z} + \frac{z}{x} \tag{23}$$

To be successful in math olympiad, it is important to know tricks such as this one.

## Summary

- The geometric mean of two positive numbers $a$ and $b$ is $\sqrt{ab}$.

[1] You will need to use the fact that $a > b > 0$ implies $1/b > 1/a$.

[2] Use $x_1 = x_2 = x_3 = x_4 = x_5 = x_6$.

- For two different positive numbers $a$ and $b$, the arithmetic mean is always bigger than the geometric mean, which in turn is always bigger than the harmonic mean.
- The arithmetic mean of $a$ and $b$ is equal to the geometric mean of $a$ and $b$, if and only if $a = b$. The similar statement can be made about the geometric mean and the harmonic mean.