Arithmetic mean and harmonic mean

Let's say a car moves with a speed of 20 km/h for an hour, then at a speed of 40 km/h for an hour. What is the average speed? It's easy. For two hours, the car moved 60 km. So, the average speed is 30 km.

Notice that the average speed will be always 30 km, as long as the amount of time the car moved at a speed of 20 km/h is the same as the amount of time it moved at a speed of 40 km/h.[1] Actually, this is an example of "arithmetic mean." The arithmetic mean of 20 and 40 is 30. More generally, the arithmetic mean of two numbers a and b is given by $(a+b)/2$.

Suppose now a car moves with a speed of 20 km/h for 100 km, then with a speed of 40 km/h for another 100 km. What is the average speed? At first glance, it may seem 30 km/h; it moved at both speeds for the same 100 km. However, notice that the car didn't move with a speed of 20 km/h and 40 km/h for the same amount of *time*. It moved for the same amount of *distance*. So, let's calculate the average speed. The car moved at a speed of 20 km/h for 5 hours (100÷20=5), and at a speed of 40 km/h for 2.5 hours (100÷40=2.5). So, it moved 200 km for 7.5 hours (5+2.5=7.5). So, by dividing 200 by 7.5, the average speed is about $26.66\cdots$ km/h. As long as the distance the car moved at a speed of 20 km/h is the same as the distance the car moved at a speed of 40 km/h, the average speed is about $26.66\cdots$ km/h. This is called the "harmonic mean." The harmonic mean of 20 and 40 is $26.66\cdots$.

Problem 1. For two positive numbers a and b, show that their harmonic mean is $2ab/(a+b)$. (Hint: Regard a and b as the speed of the car. In other words, the car moved at speed a for the distance d, then it moved at speed b for another distance d. Then, you will see that the final answer doesn't depend on d.)

Problem 2. Calculate the arithmetic mean and the harmonic mean for the following pairs of number using a calculator: (40, 50) (100, 120) (2,10) (3,5) (4,6)

If you correctly solve this problem, you will find that each of the arithmetic mean is bigger the corresponding harmonic mean. But, is the arithmetic mean of two positive numbers *always* bigger than their harmonic mean? Maybe, could somebody perhaps come up with an example that is not the case? Actually, it can be *proven* that the arithemtic mean of two positive numbers is *always* bigger than or equal to their harmonic mean, and they are equal if and only if the original two positive numbers are the same. We will show a simple proof in our article "Arithmetic mean, geometric mean, and harmonic mean." The proof uses the fact that a number squared is always non-negative. In that article, in addition to showing the proof, we will introduce geometric mean, which is defined by $\sqrt{ab}$ for two positive numbers a and b. We will see that the geometric mean is always bigger or equal to the harmonic mean,

[1] Of course, we assume here the car moved only at a speed of 20 km/h or of 40 km/h.

and always smaller or equal to the arithemtic mean. Meanwhile, you may try to figure out why the arithmetic mean is always bigger than or equal to the geometric mean yourself.

Problem 3. (Challenging!) Try to argue why the arithemetic mean of two positive numbers is always bigger or equal to their harmonic mean. Why are they equal only when the original two numbers are equal? (Hint[2])

Summary

- Arithemtic mean of two positive numbers a and b is given by $(a + b)/2$.
- Harmonic mean of two positive numbers a and b is given by the reciprocal of the arithmetic mean of $1/a$ and $1/b$.
- For two different positive numbers, their arithmetic mean is always bigger than their harmonic mean.
- The arithemtic mean and the harmonic mean of two positive numbers are the same if and only if the two numbers are the same.

[2]Without loss of generality, assume $a \leq b$. Then, recall our example of speed of car. For the example of arithmetic mean, assume the car moved with each speed for an hour. For the example of harmonic mean, assume the car moved with speed a for an hour. Then, how much amount of time did the car move with speed b? Is this amount bigger than an hour or less than an hour?