The center of mass of a triangle

See Fig. 1. You see the triangle $\triangle ABC$. D, E, and F are the midpoints of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. If you connect these midpoints with the three verticies of the triangle, they meet at one point G as in the figure. This happens for every triangle. This point is called "the center of mass of a triangle." In this article, we will prove that the three lines connecting the midpoints with the vertices indeed meet at one point by two different methods. One by a geometric method, the other by an algebraic method.

1 Geometric method

See Fig. 2. You see E and F, the midpoints of $\overline{CA}$ and $\overline{AB}$. You also see M, the intersection of $\overline{FC}$ and $\overline{EB}$.

See Fig. 3. You see D and F, the midpoints of $\overline{BC}$ and $\overline{AB}$. You also see N, the intersection of $\overline{FC}$ and $\overline{DA}$.

If we prove that M and N are the same point, we indeed prove that $\overline{FC}$, $\overline{EB}$ and $\overline{DA}$ meet at the same point. Let's show this.

See Fig. 2. again. Notice, first that $\triangle AFE$ is similar to $\triangle ABC$.

Problem 1. Why are they similar? Is it SSS or SAS or AA?

Problem 2. From this similarity, show that

$$\overline{BC} = 2\overline{FE} \tag{1}$$

As they are similar we see that $\angle AFE = \angle ABC$, which means that $\overline{FE}$ and $\overline{BC}$ are parallel to each other. Thus, we see $\angle EFM = \angle BCM$. We also have $\angle FME = \angle CMB$. Therefore, $\triangle MEF$ is similar to $\triangle MBC$ (AA similarity).

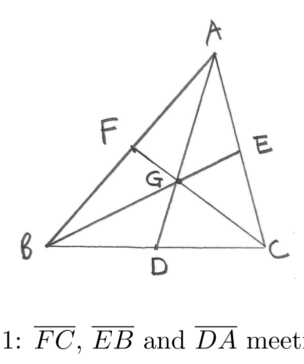

Figure 1: $\overline{FC}$, $\overline{EB}$ and $\overline{DA}$ meeting at G

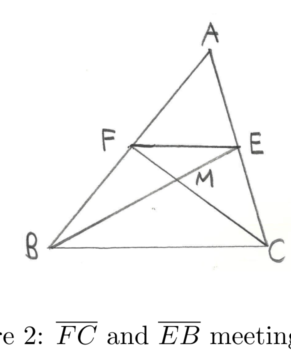

Figure 2: $\overline{FC}$ and $\overline{EB}$ meeting at M

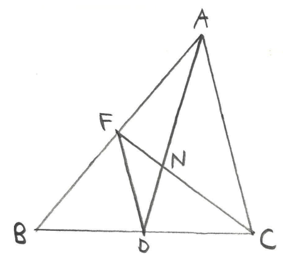

Figure 3: $\overline{FC}$ and $\overline{DA}$ meeting at N

Problem 3. Thus, show that

$$\overline{MC} = 2\overline{MF}, \qquad \overline{MB} = 2\overline{ME} \tag{2}$$

Problem 4. From (2), show that

$$\overline{MF} = \frac{1}{3}\overline{FC} \tag{3}$$

Now, see Fig. 3. The situation is similar to the one in Fig. 2. As in Fig. 2., we are connecting the two midpoints of sides with two verticies. If you take the similar steps to the ones in Fig. 2, you obtain

$$\overline{NC} = 2\overline{NF}, \qquad \overline{NA} = 2\overline{ND} \tag{4}$$

which implies

$$\overline{NF} = \frac{1}{3}\overline{FC} \tag{5}$$

Finally, compare (5) with (3). M and N are both located the same distance from F, and they are both on $\overline{FC}$. Thus, we conclude M and N are the same point.

2 Algebraic method

If we have the coordinates of the three vertices (i.e., A, B, C) of triangle, we can obtain the midpoints of each sides (i.e., D, E, F) in terms of these coordinates, and the equations for the graph for the lines connecting the vertices with the midpoints. Then, we can explicitly check whether they meet at a single point as we know how to find the coordinates that graphs intersect.

So, what should we call the coordinates of the three vertices? We could call them

$$A = (x_A, y_A), \qquad B = (x_B, y_B), \qquad C = (x_C, y_C) \tag{6}$$

Or, is there a better choice? We could rotate and align the triangle so that $y_C = y_B$. See Fig. 4. Aligning so is "Step 1." $\overline{BC}$ is horizontal (i.e., parallel to x-axis). In other words, the y-coordinates of B and C are same. Then, the coordinates of the vertices are given by

$$A = (x_A, y_A), \qquad B = (x_B, y_B), \qquad C = (x_C, y_B) \tag{7}$$

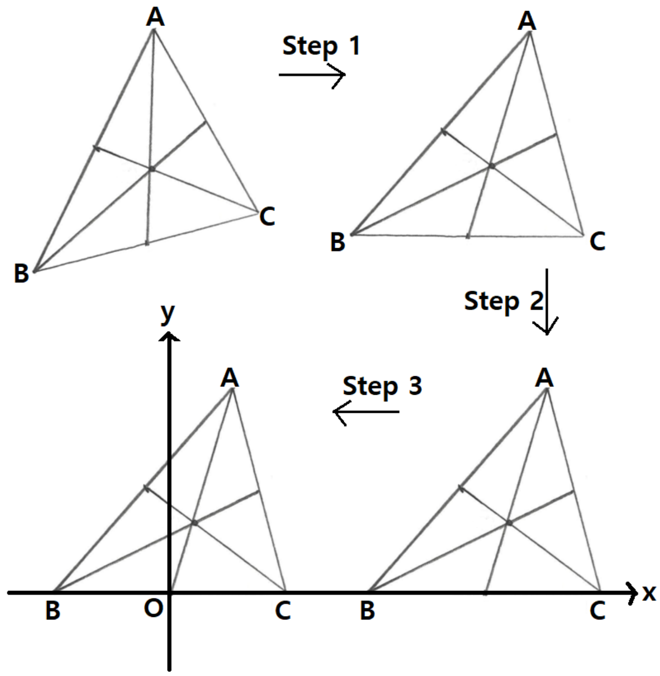

Figure 4: algebraic method

which makes the calculation slightly simplier. Could there be a even better choice? We can align the triangle such a way that $y_B = y_C = 0$. This corresponds to "Step 2" in Fig. 4. This corresponds to picking up a triangle and aligning its sides $\overline{BC}$ with x-axis. Then, the three coordinates are given by

$$A = (x_A, y_A), \qquad B = (x_B, 0), \qquad C = (x_C, 0) \tag{8}$$

which makes the calculation even simpler. Could there be a even even better choice? If we choose x_B and x_C in such a way that the coordinate of D is $(0, 0)$, the equation of the graph for $\overline{DA}$ would be of the form $y = mx + b$ where $b = 0$. This corresponds to "Step 3" in Fig. 4. This will make the calculation even even simpler. Then, the three coordinates are given by

$$A = (x_A, y_A), \qquad B = (-x_C, 0), \qquad C = (x_C, 0) \tag{9}$$

In other words, the distances from D to B and C are same, both given by x_C. It is indeed the midpoint.

Given this, let's do actual calculation. The coordinate of E, which is the midpoint of $\overline{AC}$ is given by

$$E = \left(\frac{x_A + x_C}{2}, \frac{y_A + 0}{2}\right) = \left(\frac{x_A + x_C}{2}, \frac{y_A}{2}\right) \tag{10}$$

What is the coordinate of F? You see from (9) that the coordinate of B is given by the coordinate of C with x_C replaced by $-x_C$. Thus, the coordinate of F, the midpoint of $\overline{AB}$

is given by the coordinate of E with x_C replaced by $-x_C$. Thus,

$$F = \left(\frac{x_A - x_C}{2}, \frac{y_A}{2} \right) \tag{11}$$

Now, the equation for $\overline{BE}$ is given by

$$y = \frac{y_A/2}{(x_A + x_C)/2 - (-x_C)}(x + x_C) = \frac{y_A}{x_A + 3x_C}(x + x_C) \tag{12}$$

Similarly, the equation for $\overline{CF}$ is also given by the above equation, with x_C replaced by $-x_C$ as follows:

$$y = \frac{y_A}{x_A - 3x_C}(x - x_C) \tag{13}$$

Problem 5. Show that G, the intersection of (12) and (13) is given by

$$G = \left(\frac{x_A}{3}, \frac{y_A}{3} \right) \tag{14}$$

Given this, notice that the equation for $\overline{AD}$ is given by

$$y = \frac{y_A}{x_A} x \tag{15}$$

as it both passes $(0, 0)$ and (x_A, y_A). Now, it is very easy to check that (15) passes (14).

Problem 6. Show that

$$\overline{GA} = 2\overline{GD} \tag{16}$$

Notice that this agrees with the second equation of (4). How can we derive the other relations in (2) and (4)? In employing the algebraic method, we aligned $\overline{BC}$ with the x-axis. But, we could as well have aligned $\overline{AC}$ or $\overline{AB}$ with the x-axis. Then, we would have obtained the similar relations to (16).

3 The coordinate for the center of mass, a general case

Now, we have proved that the three lines connecting the midpoints and the vertices intersect at a single point. Suppose now, we want to calculate the coordinate of the point called "the center of mass" in terms of the coordinates of arbitrarily given verticies of the triangle. In other words, we are now dealing with (6) instead of (9). Do we need to calculate all over again? No. If we use the lesson (16), the thing is much easier. First, D is given by

$$D = (x_D, y_D) = \left(\frac{x_B + x_C}{2}, \frac{y_B + y_C}{2} \right) \tag{17}$$

Notice now that (16) means

$$\overline{GD} = \frac{1}{3}\overline{DA} \tag{18}$$

From D to A, the change of coordinates is given by

$$\Delta x = x_A - \frac{x_B + x_C}{2}, \qquad \Delta y = y_A - \frac{y_B + y_C}{2} \tag{19}$$

As G is located the third way from D to A, the coordinate of G is given by

$$G = \left(x_D + \frac{\Delta x}{3}, y_D + \frac{\Delta y}{3} \right) = \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3} \right) \tag{20}$$

Thus, we see that the center of mass of a triangle is given by the average of the coordinates of the vertices of the triangle.

Of course, if our original objective was to find (20) instead of showing that the lines connecting the midpoints and vertices meet at one point, we could have begun with (6) directly, without taking the step 1, 2, 3 in Fig. 4, and the additional steps in (17), (18), (19) and (20).

4 Discussions and conclusions

In this article, we have seen that the same problem can be solved by several different methods, some of which are easier than the others. In particular, in case of the algebraic methods, we have seen that the careful choice of coordinates made the calculation much easier. Mathematics is not about boring calculations. You have to think hard to come up with the correct ways to solve problems, and if you find one, you are lucky if it is an easier one than the other ones. Also, it is not in vain to find a new proof for the same problem, because it can be valuable as its own, or it could be simpler than the earlier proof. Moreover, perhaps other unsolved problems can be solved in a similar way to your new proof.

As we will see in later articles, Johannes Kepler found Kepler's first law, second law, and third law, which describe the orbits of planets around the Sun from careful data analysis of the positions of planets. Sir Isaac Newton showed that his universal law of gravitation could explain all these three laws. His proofs were geometric ones, just like the ones in Section 2 of our article. Several generations later, physicists came up with algebraic proofs, just like the ones in Section 3 of our article. These days, only students and scholars interested in history of physics look into Newton's geometric proofs. All the other students and scholars only learn the algebraic proof these days.

Similarly, Archimedes calculated the volume and area of cones, and spheres by using geometric methods, which are quite difficult (at least, for me). Now, any clever high school students can calculate the volume and area of cones, and spheres by using algebraic method.

Thus, we indeed see that the Cartesian coordinate invented by Descartes greatly contributed to the advance of math and physics by providing algebraic approaches to geometric problems. More and more revolutions such as the introduction of the Cartesian coordinate are occuring in math and physics. I hope many more occur and more often, so that we have more tools to approach mathematical problems or physics problems.

Summary

- If you connect the midpoints of the three sides of a triangle with the three vertices of triangle, they meet at a point.

- This point is called “the center of mass” of a triangle.
- If the coordinates of the three vertices are given by

$$A = (x_A, y_A), \qquad B = (x_B, y_B), \qquad C = (x_C, y_C)$$

the coordinate of the center of mass is given by their average, namely,

$$\left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3} \right)$$