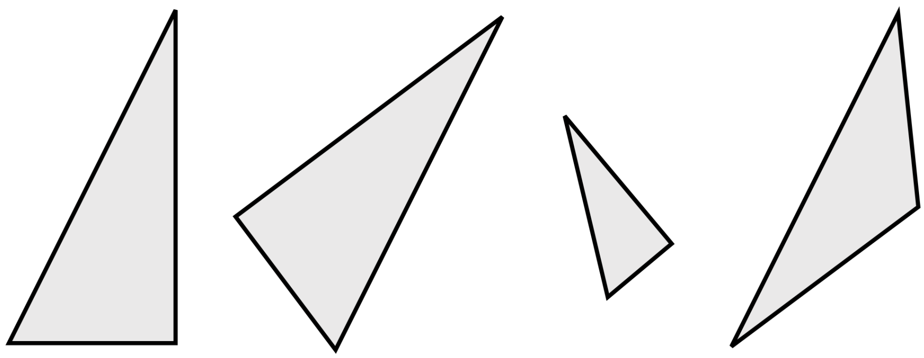
Congruence of triangle

According to Wikipedia, two figures are "congruent," if they have the same shape and size, or if one has the same shape and size as the mirror image of the other. In other words, two figures on a piece of paper are congruent, if they can be cut out and be overlapped each other completely. Turning the paper over is allowed. Two figures are "similar," if they have the same shape, of if one has the same shape as the mirror image of the other. See the figure below.



The two left triangles are congruent, and the third is similar to both of them. However, the last one is neither congruent nor similar to the other ones.

Problem 1. If two figures are congruent, are they similar as well? Explain.

There are simple ways to check whether two triangles are congruent. We list three of them.

First, SSS (Side-Side-Side). See Fig. 1. If the three sides of $\triangle ABC$ (i.e. $\triangle$ here

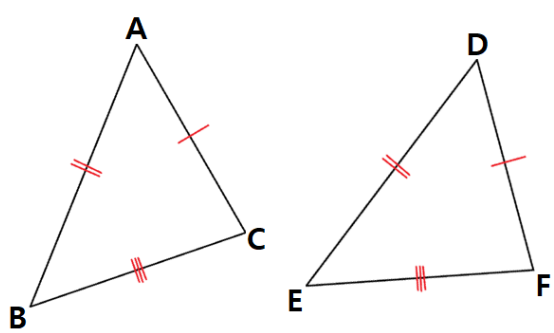


Figure 1: SSS

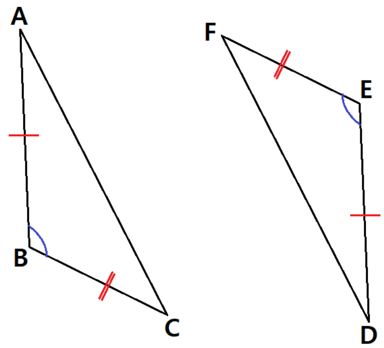


Figure 2: SAS

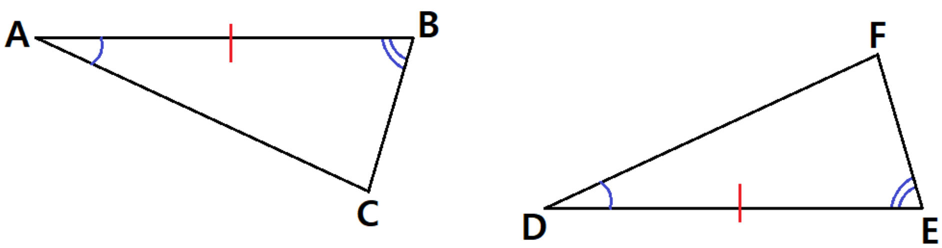


Figure 3: ASA

means triangle) have the same lengths as the three sides of $\triangle DEF$, $\triangle ABC$ and $\triangle DEF$ are congruent. It is easy to see this, if you think about how you draw a triangle when you are provided with the lengths of three sides. You are guaranteed to draw the same triangle once you are provided with the same three sides. Notice that the congruency implies $\triangle DEF$ have the same angles as the three angles of $\triangle ABC$. ($\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$)

Second, SAS (Side-Angle-Side). See Fig. 2. If two sides of $\triangle ABC$ and the angle between the two sides are the same as the two sides of $\triangle DEF$ and the angle between the two sides, $\triangle ABC$ and $\triangle DEF$ are congruent. Again, it is easy to see this, if you think about how you draw a triangle when you are provided with SAS. Again, the congruency of the two triangles imply that all of the angles and sides of $\triangle DEF$ are the same as $\triangle ABC$, not just the ones that we initially required. (i.e., we additionally obtain $\overline{AC} = \overline{DF}$, $\angle A = \angle D$, $\angle C = \angle F$.)

Third, ASA (Angle-Side-Angle). See Fig. 3. If one side of $\triangle ABC$ and the two angles adjacent to it are same as one side of $\triangle DEF$ and the two angles adjacent to it, $\triangle ABC$ and $\triangle DEF$ are congruent. Again, it is easy to see this, if you think about how you draw a triangle when you are provided with ASA. Again, the congruency implies that we additionally obtain $\overline{AC} = \overline{DF}$, $\overline{BC} = \overline{EF}$, and $\angle C = \angle F$. Now, we will apply the congruency of triangle to parallelogram. See Fig. 4. A parallelogram $ABCD$ satisfies the following four properties:

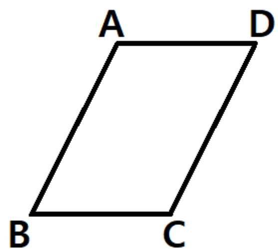


Figure 4: a parallelogram

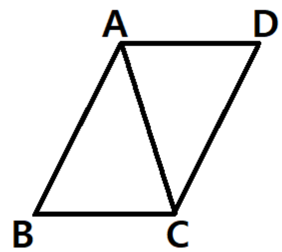


Figure 5: adding a line $\overline{AC}$

- $\overline{AB} = \overline{CD}$
- $\overline{AD} = \overline{BC}$
- $\overline{AB}$ is parallel to $\overline{CD}$.
- $\overline{AD}$ is parallel to $\overline{BC}$.

Problem 1. Some of the properties above are redundant. You will be asked to show that the third and fourth properties are automatically satisfied, if the first and second properties are satisfied. To this end, See Fig. 5. We add an additional line $\overline{AC}$, which immensely helps our analysis. Assuming $\overline{AB} = \overline{CD}$ and $\overline{AD} = \overline{CB}$, show that $\angle CAB = \angle ACD$, which implies the third property and $\angle DAC = \angle BCA$, which implies the fourth property. (Hint[1])

Problem 2. You will be asked to show that the first and the second properties are automatically satisfied, if the third and fourth properties are satisfied. Assuming the third property, which implies $\angle CAB = \angle ACD$, and the fourth property, which implies $\angle DAC = \angle BCA$, show that $\overline{AB} = \overline{CD}$, and $\overline{AD} = \overline{BC}$.

Problem 3. You will be asked to show that the first and the third properties are automatically satisfied, if the second and fourth properties are satisfied. Assuming $\overline{AD} = \overline{BC}$ and the fourth property, which implies $\angle DAC = \angle BCA$, show that $\overline{AB} = \overline{CD}$ and $\angle CAB = \angle ACD$, which implies the third property.

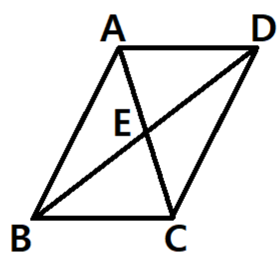


Figure 6: the diagonals of a parallelogram bisect each other

[1] Show that $\triangle ABC$ is congruent to $\triangle CAD$.

Problem 4. In this problem, you will be asked to show that the diagonals of a parallelogram bisect each other. See Fig. 6. Show that $\overline{AE} = \overline{CE}$ and $\overline{BE} = \overline{DE}$. (Hint[2])

Summary

- Two figures are "congruent," if they have the same shape and size, or if one has the same shape and size as the mirror image of the other.
- Two figures are "similar," if they have the same shape, of if one has the same shape as the mirror image of the other.
- Two triangles are congruent if one of the following three criteria is satisfied: SSS (Side-Side-Side), SAS (Side-Angle-Side), ASA (Angle-Side-Angle).

(The figure is from `https://commons.wikimedia.org/wiki/File:Congruent_non-congruent_triangles.svg`)

[2] Show that $\triangle ADE$ is congruent to $\triangle CBE$.