Convergence and divergence of series

Suppose you want to calculate

$$\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\cdots \tag{1}$$

Of course, the answer will get bigger as you add more terms, as each term is bigger than zero, and therefore contributes to the sum. Nevertheless, let's sum the first several terms and find a pattern.

If you sum the first two terms, you get

$$\frac{1}{2^1}+\frac{1}{2^2}=\frac{3}{4}=1-\frac{1}{4}=1-\frac{1}{2^2} \tag{2}$$

If you sum the first three terms, you get

$$\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}=\frac{7}{8}=1-\frac{1}{8}=1-\frac{1}{2^3} \tag{3}$$

If you sum the first four terms, you get

$$\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=\frac{15}{16}=1-\frac{1}{16}=1-\frac{1}{2^4} \tag{4}$$

So we see a pattern. We can predict that we get the following if we sum the first n terms:

$$\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n} \tag{5}$$

Of course, this is just a guess, but it's actually a correct one, and you will be able to prove this after reading the next article.

Notice that the more terms you add, the closer the sum gets to 1. This is so, because the difference between the sum of the first n terms and 1 is $1/2^n$, which is smaller for bigger n. For example, when $n = 10$, the difference is $1/1024$, and when $n = 20$, the difference is $1/1048576$, which is much smaller than $1/1024$ Actually as the difference $1/2^n$ can be arbitrarily close to 0 (pick just a big enough n), we see that as you add more terms the sum can be arbitrarily close to 1. In other words, as you add more terms, the sum approaches 1. Therefore, we can say the sum (1) is 1. Using the notation we introduced in the last article, we can re-express this statement as

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=1 \tag{6}$$

We say that the series converges to 1.

Let me show you another example of a series that converges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} \cdots \tag{7}$$

To obtain its explicit value, notice that it can be re-expressed as

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} \cdots \tag{8}$$

To gain some insight, let's calculate the above sum up to $n = 5$.

$$\begin{aligned} \sum_{n=1}^{5} \frac{1}{n} - \frac{1}{n+1} &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} \\ &= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) - \frac{1}{5} \\ &= \frac{1}{1} + 0 + 0 + 0 - \frac{1}{5} = 1 - \frac{1}{5} = \frac{4}{5} \end{aligned} \tag{9}$$

Similarly, if we calculate the above sum up to $n = 1000$, we have

$$\sum_{n=1}^{1000} \frac{1}{n} - \frac{1}{n+1} = 1 + 0 + 0 + \cdots + 0 - \frac{1}{1000} = 1 - \frac{1}{1000} = \frac{999}{1000} \tag{10}$$

Now, it is easy to see that the more number of terms you add, the closer the sum approaches 1. Thus, the sum converges to 1.

Problem 1. Calculate the following series. (Hint[1])

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \cdots \tag{11}$$

However, not all series converge. For example, consider the following series:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + \cdots \tag{12}$$

As you add up bigger numbers, the series keeps just getting bigger and bigger, surpassing any big numbers. Therefore, we cannot give a fixed, well-defined value to this sum.[2] In cases such as this in which a series doesn't converge, we say the series "diverges."

[1] Use $\frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}$. i.e., $\frac{2}{1\cdot 3} = \frac{1}{1} - \frac{1}{3}$ and $\frac{2}{3\cdot 5} = \frac{1}{3} - \frac{1}{5}$ and so on.

[2] Strictly speaking, this statement is not true. A well-defined value to this sum can be given, and if you are interested, you can read our later article "$1+2+3+4+\cdots = -1/12''$. Nevertheless, I never knew that a well-defined value could be assigned to this sum until I began to study string theory. I can safely say that even most math undergraduate students do not know this fact. Therefore, at a high school or undergraduate level, it is just simpler to say that "we cannot give a fixed, well-defined value to this sum."

Now, notice why (12) didn't converge. It's because the terms you added get bigger and bigger. For the series to converge, the terms you add must get closer and closer to 0. Otherwise, the sum to which the series converges cannot have a unique value. For example, consider the series:

$$\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+1-1+1+\cdots \tag{13}$$

You get -1 if you sum the first odd number of terms, while you get 0 if you sum the first even number of terms. As this series doesn't converge, it diverges.

However, the fact the terms you add get closer to closer to 0 is not a "sufficient" condition for the series to converge, even though it is a "necessary" one.[3] In other words, if the terms you add get closer to closer to 0, that doesn't guarantee that the series converges. For example, consider the following series[4]

$$S = \sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots \tag{14}$$

Certainly, the terms you add get smaller and smaller. However, notice

$$\begin{aligned}
&1+\frac{1}{2} = 1+\frac{1}{2}\\
&\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}\\
&\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} = \frac{1}{2}\\
&\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16} > \frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16} = \frac{8}{16} = \frac{1}{2}\\
&\frac{1}{17}+\frac{1}{18}+\cdots+\frac{1}{32} > \frac{1}{32}+\frac{1}{32}+\cdots+\frac{1}{32} = \frac{16}{32} = \frac{1}{2}\\
&\cdots\cdots\\
&\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\cdots+\frac{1}{2^m} > \frac{1}{2^m}+\frac{1}{2^m}+\cdots+\frac{1}{2^m} = \frac{2^{m-1}}{2^m} = \frac{1}{2}
\end{aligned}$$

If we sum up the above inequalities, we get

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m} > 1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} = 1+\frac{m}{2} \tag{15}$$

In this inequality, we need to consider the case $m = \infty$, so that the value 2^m is maximized. Plugging in this value for m, the right-hand side is infinite. This shows that (14) is infinite. Therefore, the series diverges.

[3] "Sufficient" conditions and "necessary" conditions are logical jargon widely used in math and science.

[4] I first learned this example from the Korean edition of the Japanese mathematics book "Mathematics you grow on your own" by Koji Shiga, but it is actually first discovered by the French philosopher Nicole Oresme in the 14th century.

Now, let us introduce alternating series. An alternating series is of the following form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n \tag{16}$$

where $a_n > 0$. In other words, if the nth term you add is positive, the $(n+1)$th term you add is negative, and if the nth term you add is negative, the $(n+1)$th term you add is positive.

Let me give you an example of an alternating series.

$$T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \cdots \tag{17}$$

You indeed see here that the sign of the terms you are adding is constantly alternating. Now, I will show that the above series converges. Notice

$$T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \left(\frac{1}{9} - \frac{1}{11}\right) + \left(\frac{1}{13} - \frac{1}{15}\right) + \left(\frac{1}{17} - \frac{1}{19}\right) + \cdots \tag{18}$$

As each term in the parenthesis is positive, we conclude

$$T > 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \tag{19}$$

On the other hand, we have

$$T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \left(\frac{1}{11} - \frac{1}{13}\right) - \left(\frac{1}{15} - \frac{1}{17}\right) - \cdots \tag{20}$$

As each term in the parenthesis is positive, we conclude

$$T < 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \tag{21}$$

Combining the above inequality with (19), we obtain

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} < T < 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \tag{22}$$

Similarly, we can obtain

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{9999} < T < 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{9999} + \frac{1}{10001} \tag{23}$$

In other words, T is always between the sum up to nth term and the sum up to $(n+1)$th term; T falls between the narrow range of size $(n+1)$th term. In the above case, $(n+1)$th term is $1/10001$. As we consider bigger and bigger n, we can make the range arbitrarily narrower and narrower. Therefore, the sum converges to a certain value.

Actually, even though we will not show it, it turns out that $T = \pi/4$. You may find it somewhat surprising that the exact sum is expressible in terms of π (i.e. circumference to diameter ratio of a circle), even though this series doesn't seem to have anything to do with a circle. Nevertheless, in mathematics, one often encounters π in cases that apparently have nothing to do with a circle.

Problem 2. Show

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2 \tag{24}$$

by noticing the following.[5]

$$\begin{aligned}
&1 = 1 \\
&\frac{1}{2^2} + \frac{1}{3^2} < \frac{1}{2^2} + \frac{1}{2^2} = \frac{2}{2^2} = \frac{1}{2} \\
&\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} = \frac{4}{4^2} = \frac{1}{4} = \frac{1}{2^2} \\
&\frac{1}{8^2} + \frac{1}{9^2} + \cdots + \frac{1}{15^2} < \frac{1}{8^2} + \frac{1}{8^2} + \cdots + \frac{1}{8^2} = \frac{8}{8^2} = \frac{1}{8} = \frac{1}{2^3} \\
&\cdots\cdots
\end{aligned}$$

You will need to use

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2 \tag{25}$$

which can be easily deduced from (6). This shows that (24) converges and the sum is smaller than 2. Actually, even though we will not show it, the exact sum is given by $\dfrac{\pi^2}{6}$.

Summary

- If a series approaches a certain value, as the sum is being evaluated for more and more terms, we say a series "converges." Otherwise, we say a series "diverges."
- For the series to converge, the terms you add must get closer and closer to 0. However, this doesn't guarantee that the series converges.

[5] This problem is also from Koji Shiga's book.