Dealing with the denominator

In this article, we introduce several tricks to deal with the denominator.

How can we simplify the following formula?

$$\frac{1}{x} - \frac{1}{x+1} \tag{1}$$

The denominators are not the same, so it is not that easy. However, in elementary school, you already learned how to deal with such cases. You multiply numerator and denominator by appropriate number to make common denominators. For example,

$$\frac{1}{2} + \frac{1}{3} = \frac{1\times 3}{2\times 3} + \frac{1\times 2}{3\times 2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} \tag{2}$$

Similarly, we have

$$\frac{1}{x} - \frac{1}{x+1} = \frac{x+1}{x(x+1)} - \frac{x}{x(x+1)} = \frac{1}{x(x+1)} \tag{3}$$

Problem 1. Simplify the following.

$$\frac{1}{x-1} - \frac{1}{x-2} = ?, \qquad \frac{1}{x-2} + \frac{2}{x+2} = ?$$

Problem 2. Show the following. This result turns out to be useful later when we discuss Bohr's model of atom.

$$\frac{1}{(n-1)^2} - \frac{1}{n^2} = \frac{2n-1}{(n-1)^2 n^2}$$

Problem 3. Simplify the following.

$$\frac{x}{(x+1)^2} - \frac{1}{x+1} = ?$$

Consider now the following expression.

$$\frac{3}{\sqrt{5}} \tag{4}$$

You see that the denominator is not a rational number. (If you do not know what a rational number is, please read our essay "$\sqrt{2}$ as irrational number.") Can we turn it into a rational number? Yes. Like this:

$$\frac{3}{\sqrt{5}} = \frac{3\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \frac{3\sqrt{5}}{5} \tag{5}$$

Now, the denominator is a rational number. Another example:

$$\frac{4}{3\sqrt{2}} = \frac{4\sqrt{2}}{3\sqrt{2}\sqrt{2}} = \frac{4\sqrt{2}}{3\times 2} = \frac{2\sqrt{2}}{3} \tag{6}$$

Problem 4. Rationalize the denominators of following numbers.

$$\frac{\sqrt{2}}{\sqrt{3}} = ?, \qquad \frac{4}{\sqrt{2}} = ?, \qquad \frac{6}{5\sqrt{3}} = ?$$

Consider the following number. Is there a way to rationalize the denominator?

$$\frac{3}{4+\sqrt{2}} \tag{7}$$

The trick is to multiply the numerator and the denominator by $(4-\sqrt{2})$ as follows.

$$\frac{3}{4+\sqrt{2}} = \frac{3(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})} = \frac{3(4-\sqrt{2})}{4^2-(\sqrt{2})^2} = \frac{12-3\sqrt{2}}{16-2} = \frac{12-3\sqrt{2}}{14} \tag{8}$$

Other examples:

$$\frac{3}{4-2\sqrt{2}} = \frac{3(4+2\sqrt{2})}{(4-2\sqrt{2})(4+2\sqrt{2})} = \frac{3(4+2\sqrt{2})}{4^2-(2\sqrt{2})^2} = \frac{12+6\sqrt{2}}{8} = \frac{6+3\sqrt{2}}{4} \tag{9}$$

$$\frac{2}{\sqrt{2}+1} = \frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{2(\sqrt{2}-1)}{2-1} = 2(\sqrt{2}-1) = 2\sqrt{2}-2 \tag{10}$$

Problem 5. Rationalize the denominators of following numbers.

$$\frac{2}{2+\sqrt{2}} = ?, \qquad \frac{1}{3-\sqrt{2}} = ?, \qquad \frac{\sqrt{2}}{2-\sqrt{2}} = ?$$

$$\frac{2}{2+3\sqrt{2}} = ?, \qquad \frac{1}{3-2\sqrt{2}} = ?, \qquad \frac{\sqrt{2}}{3\sqrt{2}-2} = ?$$

We can also rationalize a slightly different form of denominator. For example,

$$\frac{3}{\sqrt{6}-\sqrt{2}} = \frac{3(\sqrt{6}+\sqrt{2})}{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})} = \frac{3(\sqrt{6}+\sqrt{2})}{4} \tag{11}$$

Problem 6. Rationalize the following denominators.

$$\frac{3}{\sqrt{5}-\sqrt{2}} = ?, \qquad \frac{2}{\sqrt{6}-\sqrt{2}} = ?, \qquad \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}} = ?$$

Problem 7. Simplify the following by rationalizing the denominator.

$$\frac{a-b}{\sqrt{a}-\sqrt{b}} = ?,$$

Problem 8. Prove the following. (Assume $a > 0$) These relations are useful in our article "Quadratic equation."

$$\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = \frac{\sqrt{b^2-4ac}}{2a}$$

Problem 9. Read our essay "$\sqrt{2}$ as irrational number." Show that numbers of the form $a + b\sqrt{3}$ where a and b are rational numbers are closed under addition, subtraction, multiplication, and division.

Problem 10. (Challenging!) Rationalize the denominator of the following number.

$$\frac{2}{1+\sqrt{2}+\sqrt{3}} = ? \tag{12}$$

Summary

- The denominator of an expression of the form

  $$\frac{a}{\sqrt{b}}$$

  can be rationalized by multiplying the numerator and the denominator by $\sqrt{b}$.

- The denominator of an expression of the form

  $$\frac{a}{b+\sqrt{c}}$$

  can be rationalized by multiplying the numerator and the denominator by $b-\sqrt{c}$.