De Morgan's laws and the contrapositive

In the last article, we introduced set theory and used it to calculate the number of students who passed either the Spanish examination or the Korean examination, given the numbers of students who passed the Spanish examination, the Korean examination, and both examinations, respectively. In this article, we will explain De Morgan's laws, and then see yet another application of set theory called the "contrapositive."

1 Universe and complement

Universe, often denoted as U is a set of all elements that one wants to consider in a given situation. In Venn diagrams, it is often denoted by the big rectangle that encloses all the circles, which are usually used to denote sets. For example, let's say that we only want to consider natural numbers. If $A = \{1, 2, 3, 4, 5\}$ and $M = \{2, 4, 6, 8\}$, we have Fig. 1.

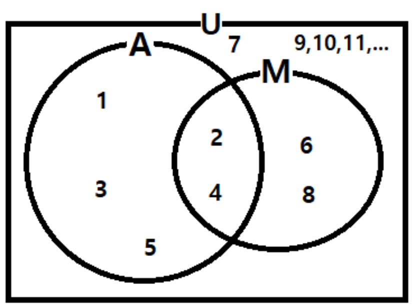

Figure 1: U is a natural number

$U \setminus A$ is often denoted as A^c, and called the "complement" of A. In Fig. 1, we see

$$A^c = \{6, 7, 8, 9, 10, 11, \cdots\} \tag{1}$$

as A^c is the set of elements that lie outside of the circle A.

Problem 1. List the 10 smallest elements of M^c.

Problem 2. List the 10 smallest elements of $(A \cap M)^c$.

Problem 3. What is $A \cap M^c$?

2 De Morgan's laws

De Morgan's laws says

$$(A \cup B)^c = A^c \cap B^c \tag{2}$$

and

$$(A \cap B)^c = A^c \cup B^c \tag{3}$$

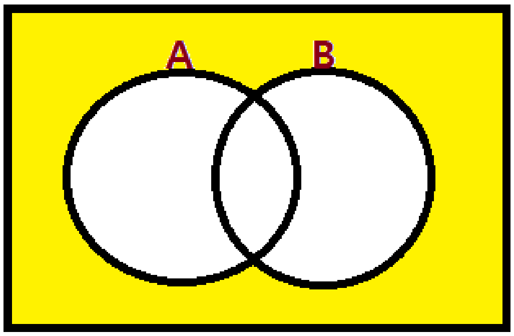

Figure 2: $(A \cup B)^c$

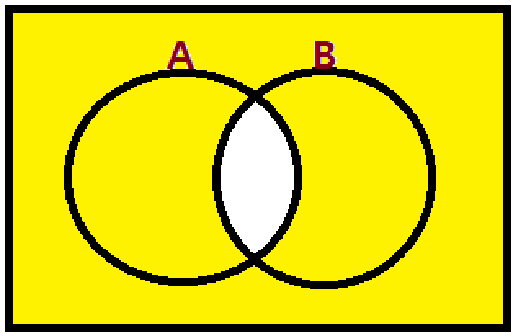

Figure 3: $(A \cap B)^c$

See Fig. 2. The white regions are $A \cup B$ and the yellow regions are $(A \cup B)^c$, which we want to prove that it is an intersection of A^c and B^c. See Fig. 4 for A^c, which is denoted by the green color. see Fig. 5 for B^c, which is also denoted by the green color. Now, it is easy to check that the region that belongs to the green region in Fig. 4 and to the green region in Fig. 5 at the same time is given by the yellow region in Fig. 2.

See Fig. 3. The white regions are $A \cap B$ and the yellow regions are $(A \cap B)^c$, which we want to prove that it is an union of A^c and B^c. See Fig. 4 for A^c, which is denoted by the green color. see Fig. 5 for B^c, which is also denoted by the green color. Now, it is easy to check that the region that belongs either to the green region in Fig. 4 or to the green region in Fig. 5 at the same time is given by the yellow region in Fig. 3.

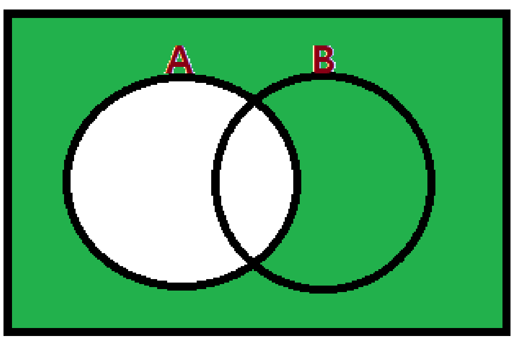

Figure 4: A^c

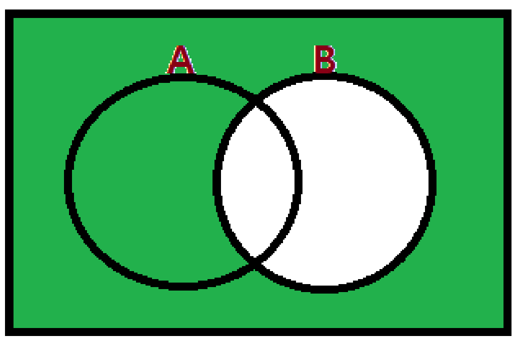

Figure 5: B^c

3 The contrapositive

If x is a human, x is an animal. This is so because all humans are animals. Let's express this statement by the language of mathematics. Let's call the set of human H, and the set of animal A. Then, if $x \in H$, then $x \in A$. In other words, we have $H \subset A$.

Let's consider a similar self-evident statement. If x is not an animal, x is not a human. Expressing this statement by the language of mathematics, we have, if $x \notin A$, then $x \notin H$. In other words, if $x \in A^c$, then $x \in H^c$. Put it differently, $A^c \subset H^c$.

We just seemed to prove that if $H \subset A$ then $A^c \subset H^c$. Let's check this again by drawing Venn diagrams.

$H \subset A$ means H is included in A as in Fig. 6. Given this, let's draw H^c and A^c. See Fig. 7 and Fig. 8. Now, notice that all the yellow region in Fig. 8 is included in the yellow region in Fig. 7. This implies $A^c \subset H^c$. This completes the proof that $H \subset A$ implies $A^c \subset H^c$. Of course, this is satisfied more generally. For any arbitrary set B, C that satisfies $B \subset C$ always satisfies $C^c \subset B^c$.

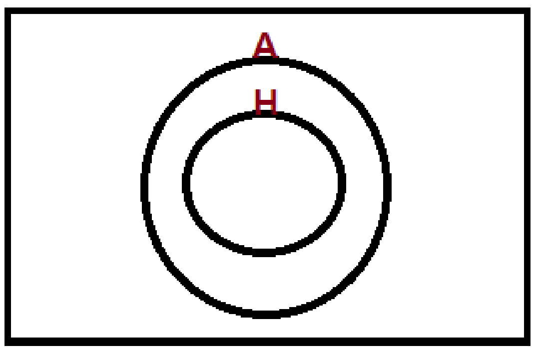

Figure 6: $H \subset A$

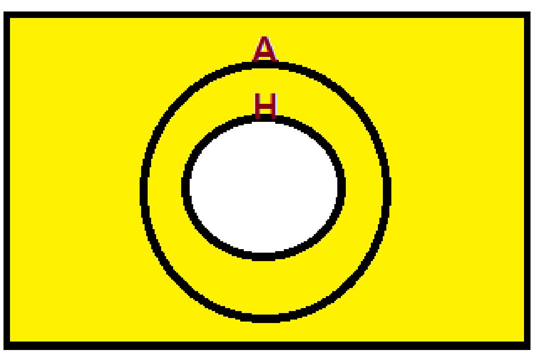

Figure 7: H^c

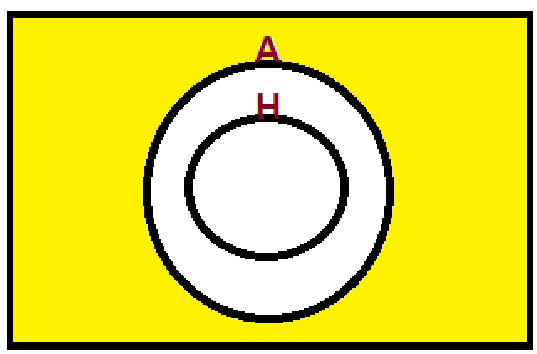

Figure 8: A^c

Now, let's introduce some terminologies. Let's consider the following statement:

$$\text{if } x \in A, \text{if then } x \in B \tag{4}$$

The inverse of the statement (4) is

$$\text{if } x \notin A, \text{if then } x \notin B \tag{5}$$

The converse of the statement (4) is

$$\text{if } x \in B, \text{if then } x \in A \tag{6}$$

The contrapositive of the statement (4) is

$$\text{if } x \notin B, \text{if then } x \notin A \tag{7}$$

In other words, the contrapositive is the inverse of the converse (or equivalently, the converse of the inverse).

If a statement is true, its contrapositive is always true. That's what we saw through the Venn diagrams. Now, notice that the contrapositive of (7) is (4). (i.e., the contrapositive of the contrapositive of a statement is the original statement.) This implies if (7) is true (4) is always true. In other words, if the contrapositive of a statement is true, the original statement is always true.

On the other hand, even though a statement is true, both its inverse and converse are not generally true. For example, consider the statement "if x is a human, x is an animal." Its inverse is "if x is not a human, x is not an animal." This is not true, because a dog is not a human, but it is an animal. Let's also consider the converse. The converse is if "x is an animal, x is a human." This is also not true, because a dog is an animal, but it is not a human. In other words, as long as $A \setminus H$ is not the empty set, the converse and the inverse of our statement are not true.

Finally, we can use the property of contrapositive to prove statements. Consider the following statement for an integer x.

$$\text{If } x^2 \text{ is an even number, then } x \text{ is an even number.} \tag{8}$$

Its contrapositive is

$$\text{If } x \text{ is not an even number, then } x^2 \text{ is not an even number.} \tag{9}$$

which is obviously true. Therefore, we conclude (8) is true. This method of proving statements by using the contrapositive is often used in mathematics.

Summary

- Universe, often denoted U is a set of all elements that one wants to consider in a given situation.
- The complement of A meaning $U \setminus A$, is often denoted as A^c.
- De Morgan's laws are $(A \cup B)^c = A^c \cap B^c$, and $(A \cap B)^c = A^c \cup B^c$.

- If $A \subset B$, then $B^c \subset A^c$.
- Consider a statement, if $x \in A$, then $x \in B$. Its contrapositive is given by if $x \notin B$, then $x \notin A$. If a statement is true, its contrapositive is always true. Similarly, if the contrapositive a statement is true, the original statement is always true.