Distances between points and equations for circles

What is the distance between point (x_1, y_1) point and (x_2, y_2)? See Fig.1. The differences between the x-coordinates is $(x_2 - x_1)$ and the difference between the y-coordinates is $(y_2 - y_1)$. Therefore, using the Pythagorean theorem, the distance d between the two points is given as follows:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

Using this, we can obtain the equation for a circle. Remember that a circle is a collection of points whose distance to the center of the circle is constant. See Fig.2. The center of the circle is (x_0, y_0) and the radius of the circle is r. Also, we know that (x, y) is a point on the circle. As its distance to the center (x_0, y_0) is r, we have:

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{2}$$

In other words,

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{3}$$

This is the equation for a circle.

So far we have considered the case in which everything is on a 2-dimensional Cartesian coordinate system. Now we will consider a 3-dimensional case. What is the distance s between the point A (x_1, y_1, z_1) and the point B (x_2, y_2, z_2)? See Fig.3. To obtain this distance, we need to apply the Pythagorean theorem to the right triangle ABC. To do so, we need to know the distance between B and C and the distance between A and C. The

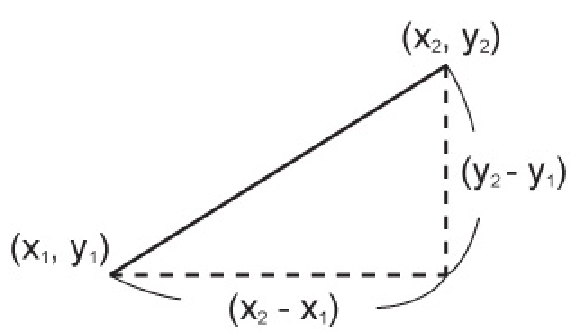

Figure 1: distance between two points

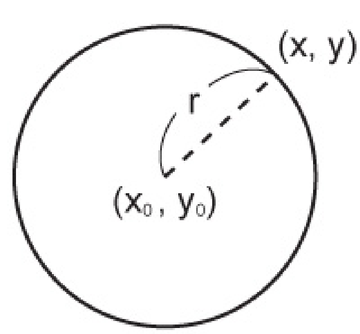

Figure 2: a circle with radius r

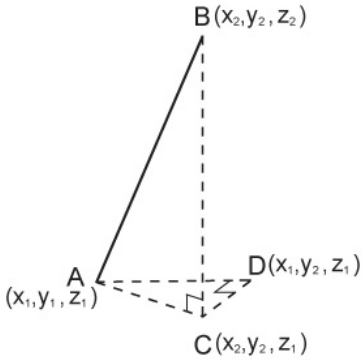

Figure 3: distance between two points in 3d

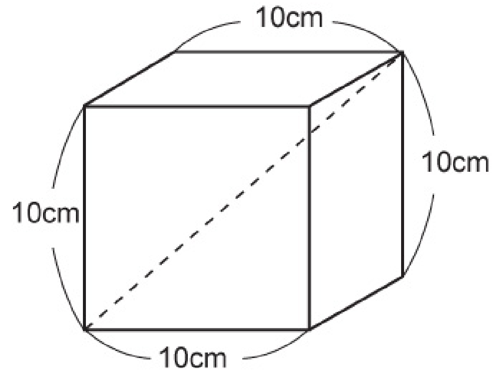

Figure 4: a cube with each side 10 cm

distance between B and C is $(z_2 - z_1)$. If we denote the distance between A and C by u, the Pythagorean theorem applied to triangle ABC says that

$$s = \sqrt{u^2 + (z_2 - z_1)^2} \tag{4}$$

To obtain u, the distance between A and C, we need to apply the Pythagorean theorem to the right triangle ADC. As D is the right angle, we have

$$u = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{5}$$

just like (1). Plugging (5) into (4), we get

$$s = \sqrt{\left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)^2 + (z_2 - z_1)^2} \tag{6}$$

In conclusion, we have:

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{7}$$

Given this, what is the equation for a sphere? Remember that a sphere is a collection of points whose distance to the center of a sphere is constant. If the radius of the sphere is r, and the center of sphere is at (x_0, y_0, z_0), the equation for such a sphere is given as follows:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{8}$$

Let me conclude this article with some comments. It is easy to imagine that if the space is somehow curved, formulas such as (2) and (7) are not satisfied. After all, in deriving (7) using the Pythagorean theorem, we assumed that the axes of Cartesian coordinate system are perpendicular to one another. However, this assumption may not be true generally, and therefore may not hold if the space is curved. Mathematicians have studied how formulas such as (2) and (7) are modified when the space is curved, and Einstein used their mathematics to construct the "theory of general relativity." According to this theory, spacetime is curved in the presence of heavy objects. As this mathematics was new and quite unknown to the physicists then, Einstein had to include introduction to it in his groundbreaking paper on the theory of general relativity. You will learn the theory of general relativity in our later article "A Relatively Short Introduction to General Relativity."

Problem 1. See Fig.4. We have a cube where each side has a length of 10 centimeters. What is the length of the diagonal of the cube denoted by the dotted line?

Summary

- The distance between point (x_1, y_1) point and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- If the center of a circle is (x_0, y_0) and its radius is r, its equation is given by

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

- The distance between point (x_1, y_1, z_1) point and (x_2, y_2, z_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- IF the center of a sphere is (x_0, y_0, z_0) and its radius is r, its equation is given by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$