Exponents, revisited

In our earlier article "Exponents," we introduced the basic concept of exponents. In this article, we will investigate some of its properties.

Problem 1. Simplify x^5x^3. Simplify $x^{2b}x^{3b}$. Simplify x^{2a}/x^a.
(Note: x^5x^3 means $x^5 \times x^3$. x^{2b} means $x^{(2b)}$)

Now, let's evaluate the following quantity:

$$(x^a)^4 = x^a x^a x^a x^a = x^{a+a+a+a} = x^{4a} \tag{1}$$

Therefore, more generally, we can write:

$$(x^a)^b = x^{ab} \tag{2}$$

In our earlier article "Exponents" you have figured out that $(-1)^a$ is 1 if a is an even number, and -1 if a is an odd number. Another way of obtaining this result is following. If a is an even number we can say $a = 2n$ for some integer n. (For example, if a is 10 n is 5.) Then, by using (2) we see that

$$(-1)^a = (-1)^{2n} = \left((-1)^2\right)^n = 1^n = 1 \tag{3}$$

Similarly, if a is an odd number $a = 2n+1$ for some integer n. (For example, if a is 5 n is 2.)

$$(-1)^{2n+1} = (-1)^{2n}(-1)^1 = 1 \cdot (-1) = -1 \tag{4}$$

There is also another property for exponents. Consider:

$$(ab)^4 = abababab = aaaabbbb = a^4b^4 \tag{5}$$

In other words, we have:

$$(ab)^n = a^n b^n \tag{6}$$

Similarly, one can show:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \tag{7}$$

Problem 2. Show the following.

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m \tag{8}$$

In this article, we only consider the cases in which exponents are integers. In the next article, we will see how this can be generalized to the cases when the exponents are not integers.

Problem 3. Evaluate the following:

$$(-2)^3 = ?, \qquad (-3)^{-3} = ?, \qquad \left(-\frac{1}{4}\right)^{-2} = ?, \qquad \left(-\frac{1}{2}\right)^{-4} = ? \tag{9}$$

Problem 4. Which of the following is equal to $\dfrac{1}{2 \times 10^6}$? Answer without using a calculator.

(a) 5×10^{-5} (b) 5×10^{-6} (c) 5×10^{-7}

Problem 5. Simplify the following:

$$x^{3a}(x^a)^{-2} = ?, \qquad x^{a+b}x^{3a-b} = ?, \qquad \frac{x^{a+2b}}{x^b} = ?$$

$$(x^{2x+3})^2 = ?, \qquad (3x)^3 = ?, \qquad \left(\frac{x}{2}\right)^4 = ?$$

$$\frac{(6x^2y^3)^2}{(2xy^2)^3} = ?, \qquad \frac{(ab)^{-1}}{1/a} = ?$$

Problem 6. Let's say that the population of a certain country on New Year's day, 2000, is 5 million, and the population grows by 7 percents every year. Notice that the population on New Year's day, 2001 is given by 5×1.07 million and the population on New Year's day, 2002 is given by $5 \times 1.07 \times 1.07$ million. Given this, use the exponent notation to express the formula for the population on New Year's day 2015, and obtain its explicit value using a scientific calculator such as the one provided by Microsoft Windows. Also, by calculating the population on New Year's day of each year beginning from 2000, obtain the year in which the population on its New Year's day exceeds 7 million for the first time. (You can easily obtain this value as it is not a distant future from 2000. However, if I asked when would be the year in which the population on its New Year's day exceeds 40 million for the first time, which is a distant future, it would not be that easy to obtain the answer using the method I suggested. However, in our later article "Logarithm," you will see there is a much simpler and faster method to obtain the answer, if we use the concept of "logarithm.")

Summary

- $(ab)^n = a^n b^n$.
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.