Exponents

Consider the following quantity:

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \tag{1}$$

It is too long: is there a simpler way to express this? There is. We write:

$$3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \tag{2}$$

where 6 here denotes the fact that 3 is multiplied 6 times. Here, we call 3 the "base" and 6 the "exponent." We pronounce 3^6 as "3 to the power of 6."

Problem 1. Express the followings using exponent notation.

$$5 \times 5 \times 5 = ? \qquad (-2) \times (-2) \times (-2) \times (-2) = ?$$

In mathematics, science and engineering we often meet cases in which the exponent is 2 or 3, so we have special ways to pronounce such expressions. For example, instead of pronouncing 5^2 as "5 to the power of 2", we often pronounce it as "5 squared." Similarly, we often pronounce 5^3 as "5 cubed."

Problem 2. What is 5 squared? What is 2 cubed?

Problem 3. Evaluate the following.

$$3^4 = ?, \qquad (-2)^3 = ?, \qquad \left(\frac{1}{3}\right)^2 = ?$$

Problem 4. In this problem, we will revisit the rule that we have learned in the last article. Notice that $(-1)^2 = 1$, $(-1)^3 = -1$, $(-1)^4 = 1$, $(-1)^5 = -1$ and so on. From this pattern, evaluate $(-1)^{2016}$ and $(-1)^{4349}$.

Problem 5. Evaluate the following.

$$10^2 = ?, \qquad 10^3 = ?, \qquad 10^4 = ? \tag{3}$$

If you correctly solved this problem you will see that 10^n has 1 followed by n numbers of 0. For example, 10^4 is 10,000 as it must be 1 followed by 4 numbers of 0.

Problem 6. What is 10^8?

Can an exponent be a negative number? For example, what is 2^{-3}? At first glance, this seems to be a senseless question, as we cannot certainly multiply a number negative times. However, it turns out that we can assign a value to a negative exponent consistently. To this end, we need to understand some properties of exponents first.

Suppose you want to calculate $5^4 \times 5^3$. We have:

$$\begin{aligned} 5^4 \times 5^3 &= (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5) && (4) \\ &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 && (5) \\ &= 5^7 && (6) \end{aligned}$$

In thinking how you have gotten the exponent 7, you will see that you just summed 4 and 3. Therefore, more generally, we can write:

$$a^b \times a^c = a^{b+c} \tag{7}$$

Similarly, suppose you want to calculate $5^5 \div 5^2$. We have:

$$\begin{aligned} \frac{5^5}{5^2} &= \frac{5 \times 5 \times 5 \times 5 \times 5}{5 \times 5} && (8) \\ &= 5 \times 5 \times 5 = 5^3 && (9) \end{aligned}$$

In thinking how you have gotten the exponent 3, you will see that you subtracted 2 from 5. Therefore, more generally, we can write:

$$\frac{a^b}{a^c} = a^{b-c} \tag{10}$$

Problem 7. What is $2^{2019} \div 2^{2016}$?

Given this, what is a^0? Certainly, multiplying something 0 times doesn't really make much sense. Nevertheless, we can assign to it a certain value that is mathematically consistent. First, notice:

$$a^d \times a^0 = a^{d+0} = a^d \tag{11}$$

In other words, if you multiply a^d by a^0, you get the same number a^d. So, we conclude:

$$a^0 = 1 \tag{12}$$

Of course, there is another way of seeing this:

$$\frac{a^d}{a^d} = a^{d-d} \tag{13}$$

As the left-hand side is 1 and the right-hand side a^0, we can conclude again that $a^0 = 1$.

Problem 8. What is 2.5^0?

Finally, negative exponent. Let's divide both sides of (12) by a^b. We get

$$\frac{a^0}{a^b} = \frac{1}{a^b} \tag{14}$$

$$a^{0-b} = \frac{1}{a^b} \tag{15}$$

$$a^{-b} = \frac{1}{a^b} \tag{16}$$

This is a very useful result. For example,

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27} \tag{17}$$

As another example,

$$\left(\frac{1}{5}\right)^{-2} = \frac{1}{\left(\frac{1}{5}\right)^2} = \frac{1}{1/25} = 25 \tag{18}$$

Problem 9. Evaluate the following.

$$4^{-2} = ?, \qquad \left(\frac{1}{2}\right)^{-3} = ?, \qquad 10^{-2} = ?, \qquad 10^{-3} = ? \tag{19}$$

Problem 10. Answer without using a calculator.

- Compare the numbers 2^{100} and 3^{100}. Which is bigger?
- Compare the numbers 2^{100} and 2^{102}. Which is bigger?
- Compare the numbers 0.5^{100} and 0.5^{102}. Which is bigger?
- Compare the numbers 0.4^{100} and 0.6^{100} Which is bigger?
- Compare the numbers 3^{-100} and 3^{-101} Which is bigger?
- Compare the numbers 0.4^{-100} and 0.6^{-100}. Which is bigger?

To conclude this article, let me ask you a question. Can an exponent be a fraction? For example, what is $9^{1/2}$? At first glance, this seems to be a senseless question, as we cannot certainly multiply a number half times. However, as we just learned that a negative exponent makes sense, it also turns out that we can assign a value to a non-integer exponent. We will talk about it in "Root, cube root, and nth root, revisited."

Summary

- $a^b \times a^c = a^{b+c}$
- $\dfrac{a^b}{a^c} = a^{b-c}$.
- $a^0 = 1$, if $a \neq 0$.