What is a function?

Let's say that there is a machine in which if you enter a number (i.e. an input), another number (could be the same or different) pops out (i.e. an output). Let's also say that the number that comes out can depend on the number you entered. This is the concept of a function. The number that pops out is a function of the number that you entered. We can express this mathematically as follows. If the number you entered was x, the number popped out is "$f(x)$." $f(x)$ means the function f of x, and is pronounced as "f x." $f(x)$ gives us the rule that defines which number comes back. For example, if $f(x) = x^2 - 1$, then $f(3) = 8$. So, if I enter 3, the machine pops out 8.

Notice that by definition, only one number pops out when one number is entered; two or three numbers do not pop out. Think about vending machines. It would be unpractical, if you get different items, even though you press the same button. In other words, it would be malfunctioning vending machine. A function is like a well-functioning vending machine. If the input is the same, the ouput is the same. Given this, we can now explain why the square root of a number can't be two numbers. The square root is a function, so if it takes one number, it should spit out one number. $\sqrt{9}$ is 3, not 3 and -3.

Sometimes no number will pop out even when a number is entered, if a function is not defined for that number. For example, if $f(x) = 1/(x-2)$, $f(2)$ is not defined, as you can never divide something by zero. As another example, if $g(x) = \sqrt{x}$, $g(-3)$ is not defined as there is no square root for a negative number.

Given this, we can now introduce the concept of a "domain." The domain of a function is a set of input variables for which the function is defined. For example, the domain of $g(x) = \sqrt{x}$ is all non-negative real numbers. (Real numbers are just ordinary numbers such as 1,-3.4,π, $-\sqrt{2}$, $\frac{3}{4}$ and so on.[1]) For another example, the domain of $f(x) = 1/(x-2)$ is all real numbers except for 2.

Let us introduce some more terminologies. The "codomain" of a function is the set in which all the outputs of a function are constrained to fall.

[1] In our later articles in the section "Complex numbers" we will introduce numbers that are not "real." You may sneak a peak at the articles there if you are impatient. However, at present, it is enough to think of real numbers as "ordinary numbers."

Usually, this is the set real number. The "image" of a function is the set of output variables. For example, the image of $f(x) = 1/(x-2)$ is all real numbers except for 0. As another example, the image of $g(x) = \sqrt{x}$ is all non-negative real numbers. According to Wikipedia, the term "range" is used to refer to either codomain or image.

In mathematics, real numbers are denoted as $\mathbb{R}$. Therefore, a function whose domain is all real numbers and whose range is all real numbers is denoted as follows:

$$f : \mathbb{R} \to \mathbb{R} \tag{1}$$

Sometimes, we consider functions whose inputs are n numbers and outputs are m numbers as follows:

$$f : \mathbb{R}^n \to \mathbb{R}^m \tag{2}$$

For example, $f(x, y, z) = (x^2 - z, x^2 - y^2 + z)$ is an example of a function $f : \mathbb{R}^3 \to \mathbb{R}^2$.

Final remark. Notice that the definition of function doesn't forbid the possibility that we can get the same output with different inputs. For example, $f(x) = x^2$ is a perfectly valid function, even though $f(2) = f(-2) = 4$.

Problem 1. Find the domain and the image of the following functions.

$$f(x) = \sqrt{x+3}, \qquad g(x) = x^2 + 2 \tag{3}$$

Problem 2. Find the domain and the image of the function $h(x) = 2/(x^2+2)$. (Hint[2])

Problem 3. Find the domain and the image of the function $f(x) = x^2 - 4x + 5$. (Hint[3])

Problem 4. Find the domain and the image of the function $g(x) = 1/(x^2 - 4x + 5)$. (Hint[4])

Problem 5. Find the domain and the image of the function $h(x) = 8/(x^2 - 4)$. (Hint[5])

Problem 6. Show that if you divide $f(x)$, a polynomial of x, by $x - \lambda$, the remainder is $f(\lambda)$. (Hint: we can write $f(x) = (x-\lambda)q(x)+r$ where $q(x)$ is the quotient and r is remainder. Plug in $x = \lambda$ to this equation.) This property has a far-reaching consequence. If $x = a$ is a solution to $f(x) = 0$ where $f(x)$ is a polynomial of x, then $x - a$ divides $f(x)$. Similarly, if $x - a$ divides $f(x)$, $x = a$ is a solution to $f(x)$. This is known as the "factor theorem" and was proved by Thomas Harriot in the 17th century. He was

[2]For the domain, notice that $(x^2 + 2)$ is never zero. For the image, use the result from Problem 1.

[3]For the image, you can complete the square to re-express the function in the form $f(x) = (x-d)^2 + e$. Then, it is easy to see that the minimum of $f(x)$ is e.

[4]For the image, use the result of Problem 3.

[5]For the domain, notice that the function is not defined when the denominator is 0. For the image, find the range of the denominator first.

an English astronomer, mathematician, ethnographer and translator who invented the symbol "$>$" and "$<$." He was also one of the main characters associated with the introduction of potato to Europe. Regarding the factor theorem, Harriot's ingenuity was that he came up with the idea to move every term on the right-hand side of an equation to its left-hand side to make the right-hand side zero. We will see the usefulness of the factor theorem in our later article "Fundamental theorem of algebra."

Problem 7. Let's say that you have a polynomial $f(x)$, which you divide by $x^2 - 6x + 5$. If the remainder is $2x - 5$, what is $f(1)$ and $f(5)$? (Hint[6])

Problem 8. Let's say that you have a polynomial $g(x)$, which you divide by $x^2 - 6x + 5$. Find the remainder in terms of $g(1)$ and $g(5)$. (Hint[7])

Summary

- Function is a machine which pops out an output, if you enter an input.
- We often denote a function f by $f(x)$, where $f(x)$ is the output for the input x.
- The "domain" of a function is a set of input variables for which the function is defined.
- The "image" of a function is the set of output variables.
- Functions whose inputs are n real numbers and outputs are m real numbers are denoted as

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

[6] We can write $f(x) = (x^2 - 6x + 5)h(x) + 2x - 5$ for some $h(x)$. Plug in $x = 1$ and $x = 5$ to this expression of $f(x)$.

[7] We can write $g(x) = (x^2 - 6x + 5)h(x) + ax + b$ where $ax + b$ is the remainder.