Shifting and reflecting graphs

In this article, we will talk about how to shift and reflect graphs. To begin with, we will first talk about how to shift and reflect points. If we move a point (x, y) by a in the positive x direction and by b in the positive y direction, the new point (x', y') is given by

$$(x', y') = (x + a, y + b) \tag{1}$$

See Fig. 1.

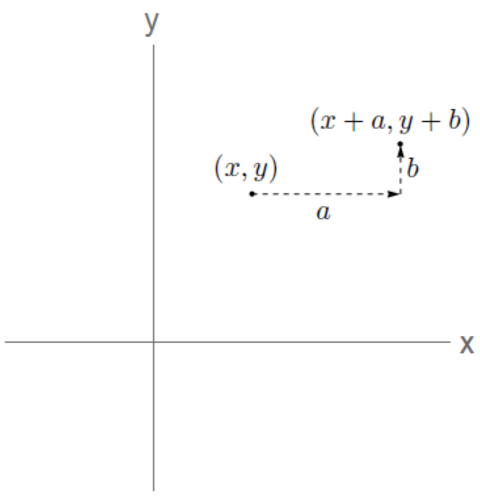

Figure 1: moving (x, y) by a in the positive x-direction and by b in the positive y-direction

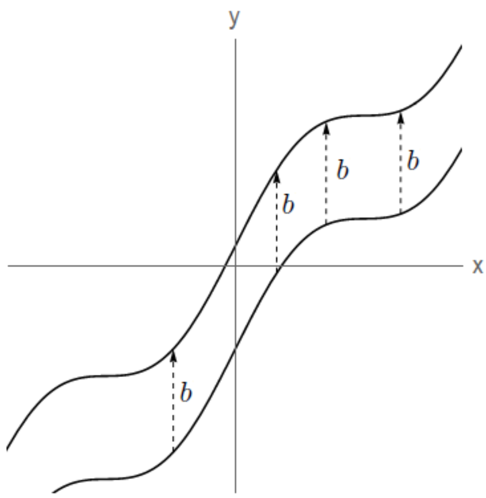

Figure 2: moving $y = f(x)$ by b in the positive y-direction

Now let's shift graphs. If we have a graph $y = f(x)$, and if we shift this graph by b in the positive y-direction, what will be the new graph? See Fig. 2. If you remember Fig. 5 of "The Cartesian coordinate system and graph" you will figure out easily that shifting a graph $y = f(x)$ by 1 in the positive y direction leads to the graph $y = f(x)+1$. In other words, shifting a graph $y = f(x)$ by b in the positive y direction leads to the new graph $y = f(x) + b$.

How about shifting a graph $y = f(x)$ by a in the positive x-direction? See Fig. 3. We have to add a to the x-coordinate of the graph. If we write $y = f(x)$ as $x = f^{-1}(y)$, the new graph is

$$x = f^{-1}(y) + a \tag{2}$$

$$x - a = f^{-1}(y) \tag{3}$$

So, the new graph is $y = f(x - a)$.

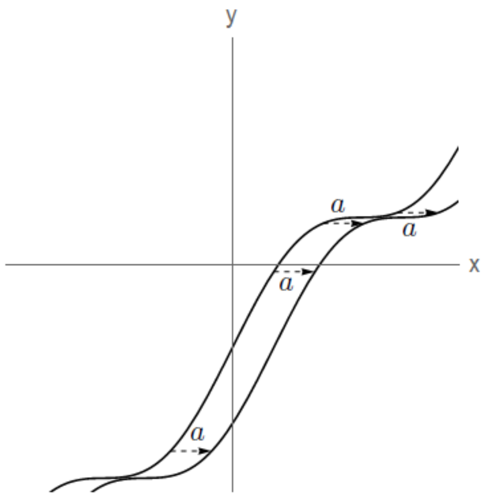

Figure 3: moving $y = f(x)$ by a in the positive x-direction

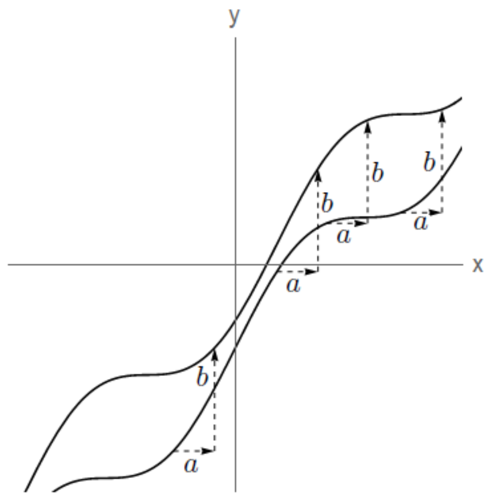

Figure 4: moving $y = f(x)$ by a in the positive x-direction, and then by b in the positive y-direction

How about shifting a graph $y = f(x)$ by a in the positive x-direction and then by b in the positive y-direction? See Fig. 4. After the first step, we have $y = f(x - a)$. The second step is adding b to the y-coordinate, which leads to

$$y = f(x - a) + b \tag{4}$$

Actually, there is another way of deriving this result. Let's say (x, y) satisfies $y = f(x)$. In other words, it is a point on the graph $y = f(x)$. Then, if we move this graph by a in the positive x-direction and by b in the positive y-direction, the new coordinate (x', y') is given in terms of (x, y) by (1), which we repeat here for convenience.

$$(x', y') = (x + a, y + b) \tag{5}$$

However, notice that this relation means

$$(x, y) = (x' - a, y' - b) \tag{6}$$

As (x, y) satisfies $y = f(x)$, we have

$$y' - b = f(x' - a) \tag{7}$$

A point (x', y') on the new, shifted graph satisfies this relation. If we relabel this point by (x, y), instead of (x', y'), we get

$$y - b = f(x - a) \tag{8}$$

which is exactly equivalent to (4). The advantage of expressing the graph by (8) instead of (4) is that it looks more "symmetrical"; x and y are on equal footing. If you move by a in the positive x-direction, and by b in the positive y-direction, you have to subtract these numbers in the coordinate and plug them back into the original relation f. No matter whether it is x-coordinate or y-coordinate. Then, you get (8). On the other hand, this equal footingness of x and y is not that apparent in the equivalent expression (4), unless you change it into the form (8). One can get the (*false*) impression that x and y are not treated equally, as you add b to the y-coordinate, while you subtract a to the x-coordinate.

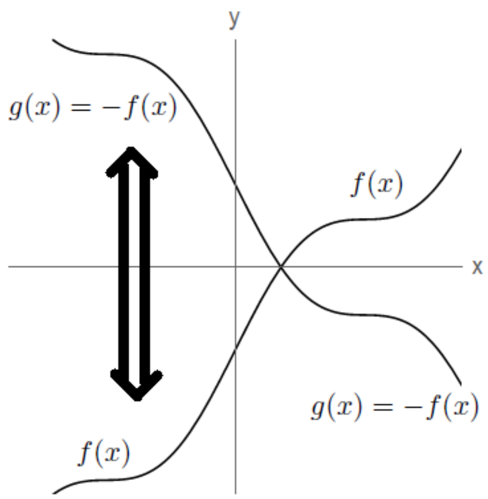

Figure 5: $g(x)$ is the reflection of $f(x)$ over the x-axis

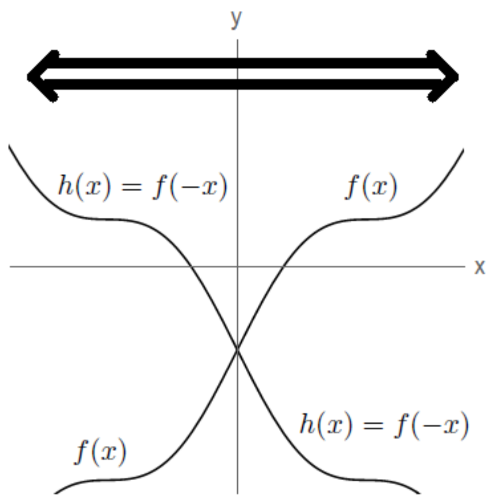

Figure 6: $h(x)$ is the reflection of $f(x)$ over the y-axis

Let's now consider reflections. What do I mean by reflection? See Fig.5. Graph $g(x)$ is the reflection of graph $f(x)$ over the x-axis. In other words, if you draw the graph $f(x)$ with ink on a paper and fold it on the x-axis, it will print $g(x)$. See Fig. 6. Graph $h(x)$ is the reflection of $f(x)$ over the y-axis. Again, if you fold $f(x)$ on the y-axis, it will "print" $h(x)$.

Let's now consider how to reflect a point over the x-axis. See Fig. 7. We reflect the point P, $(3, 4)$ over the x-axis. The distance between P and x-axis is 4. In particular, P is 4 units above the x-axis. The reflected point is 4 units below the x-axis. Therefore, the y coordinate of the reflected point is -4. As the x-coordinate of a point doesn't change as you reflect it over the x-axis, the reflected point of P is $(3, -4)$. More generally, the reflection over the x-axis sends (x, y) to $(x, -y)$.

Similarly, the reflection over the x-axis sends the point $(3,4)$ to $(-3,4)$. See Fig. 8. More generally, the reflection over the y-axis sends (x,y) to $(-x,y)$

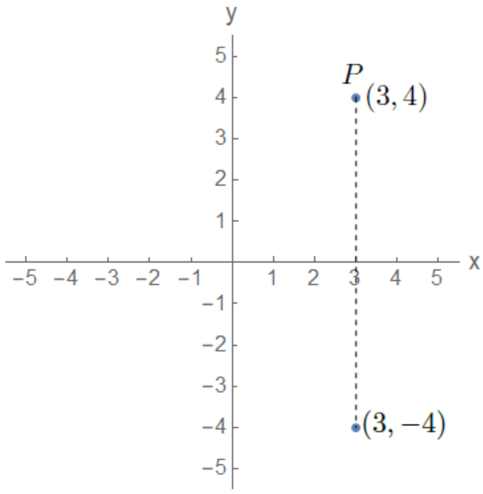

Figure 7: reflection of the point $(3,4)$ along x axis

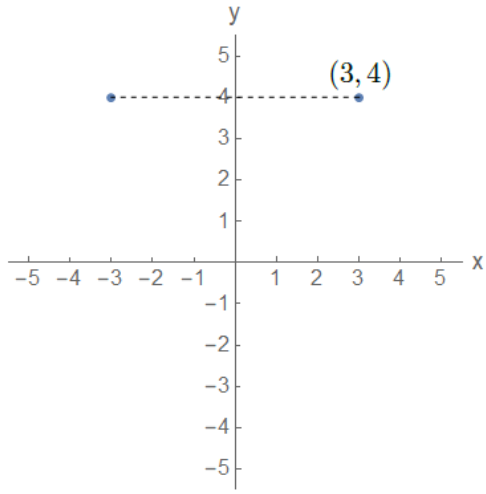

Figure 8: reflection of the point $(3,4)$ along y axis

Then, how does the equation of a graph changes under these reflections? You should be able to figure out that the reflection along the x-axis sends $y = f(x)$ to $y = -f(x)$ (or equivalently, $-y = f(x)$) and the reflection along the y-axis sends $y = f(x)$ to $y = f(-x)$. In other words, $g(x)$ in Fig. 5 is $-f(x)$, and $h(x)$ in Fig. 6 is $f(-x)$.

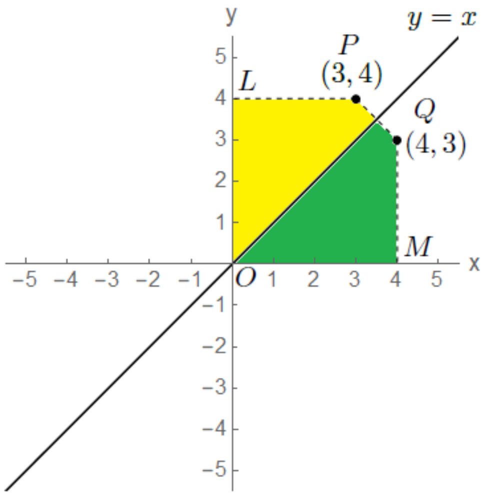

Figure 9: reflection of $(3,4)$ over the line $y = x$ is $(4,3)$

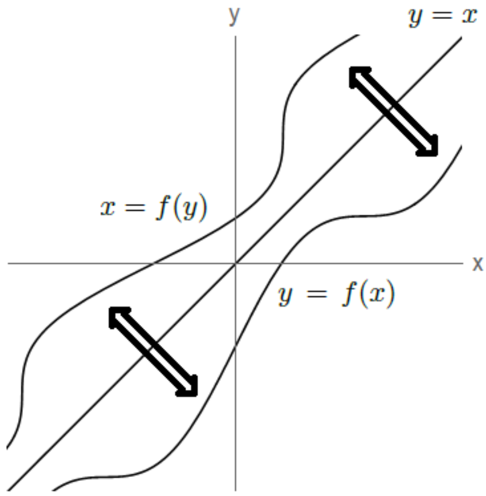

Figure 10: reflection of $y = f(x)$ over the line $y = x$ is $x = f(y)$

Finally, we will consider the reflection over the line $y = x$. See Fig. 9. We want to reflect the point P, $(3,4)$ over the line $y = x$. Then, we get Q. Notice that the distance between Q and the line $y = x$ must be equal to the distance between P and the line

$y = x$. Also, $\overline{PQ}$ meets $y = x$ with the right angle. Thus, if you fold the graph on the line $y = x$, the yellow region will meet squarely with the green region. Therefore, the length of $\overline{OM}$ will be 4, as the length of $\overline{OL}$ is 4. Thus, the x coordinate of Q is 4. Similarly, the length of $\overline{QM}$ will be 3, because the length of $\overline{LP}$ is 3. Thus, the y coordinate of Q is 3. In conclusion, the coordinate of Q is $(4, 3)$. Remember that we began with $(3, 4)$. Upon the reflection along $y = x$, the x coordinate and the y coordinate are swapped. In other words, the reflection over the line $y = x$ sends (x, y) to (y, x).

Therefore, if we reflect $y = f(x)$ over the line $y = x$, we need to swap x and y, which results in $x = f(y)$. See Fig. 10.

Summary

- If you shift (x, y) by a in the positive x-direction, and by b in the positive y-direction, you get $(x + a, y + b)$.
- If you shift $y = f(x)$ by a in the positive x-direction, and by b in the positive y-direction, you get $y - b = f(x - a)$.
- If you reflect (x, y) over the x-axis, you get $(x, -y)$.
- If you reflect $y = f(x)$ over the x-axis, you get $y = -f(x)$.
- If you reflect (x, y) over the y-axis, you get $(-x, y)$.
- If you reflect $y = f(x)$ over the y-axis, you get $y = f(-x)$.
- If you reflect (x, y) over the line $y = x$, you get (y, x).
- If you reflect $y = f(x)$ over the line $y = x$, you get $x = f(y)$.