Graphs of quadratic polynomials

In this article, we will consider the graph of the form $y = ax^2 + bx + c$. Let's start with the simplest case $b = c = 0$, and $a > 0$.

Problem 1. In Fig. 1 we have three graphs: A, B, C. They are $y = x^2, y = 2x^2, y = \frac{x^2}{2}$. Which one is which? Notice that y can be never negative for all three graphs as something squared multiplied by a positive number is always positive or 0. The minimum for y, 0 is achieved when x is 0.

Here, you see that C is sharper than B, and B is sharper than A. If you correctly solve this problem, you will find that, for positive f and g, if f is bigger than g, $y = fx^2$ is sharper than $y = gx^2$. Notice also that y increases as x goes to negative infinity or positive infinity.

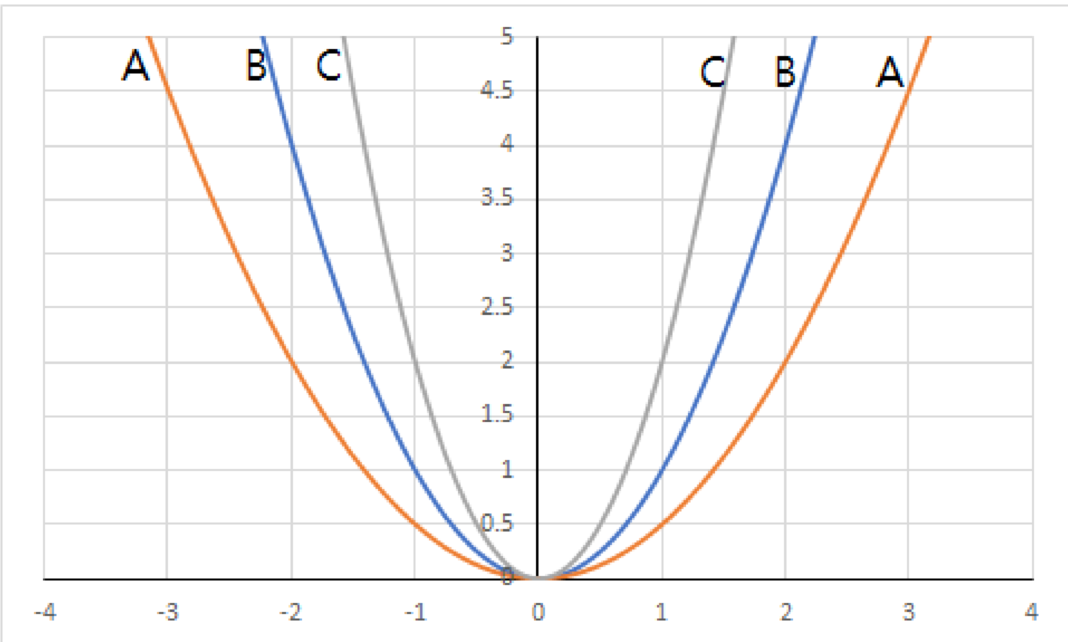

Figure 1: $y = x^2, 2x^2, \frac{x^2}{2}$

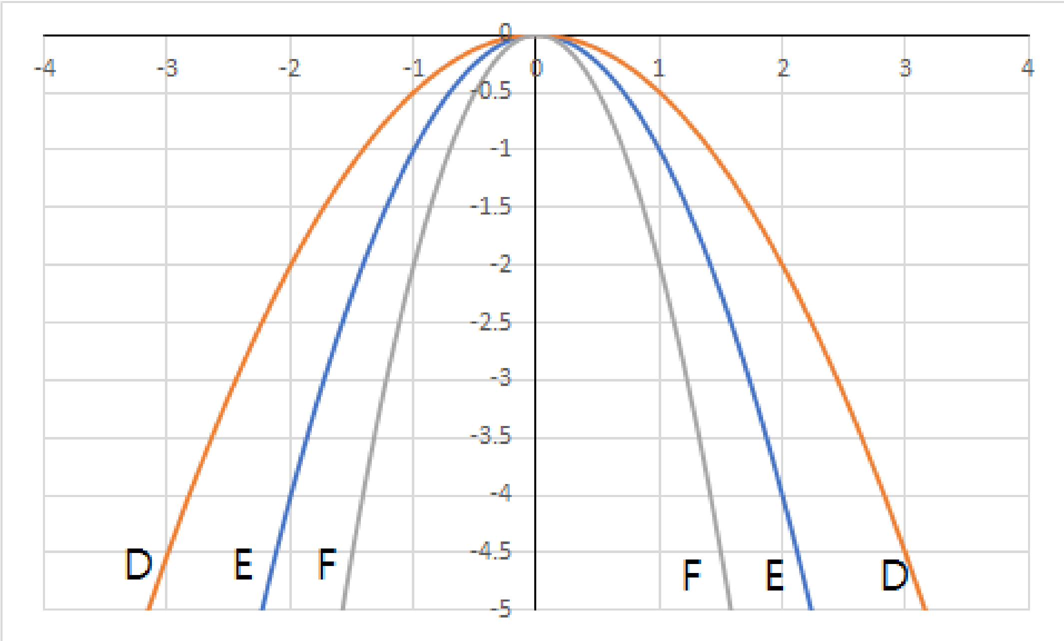

Figure 2: $y = -x^2, -2x^2, -\frac{x^2}{2}$

Now let's consider $y = ax^2$ when $a < 0$.

Problem 2. In Fig. 2 we have three graphs: D, E, F. They are $y = -x^2, y = -2x^2, y = -x^2/2$. Which one is which? Notice that y can be never positive for all three graphs as something squared multiplied by a negative number is always negative or 0. The maximum for y, 0 is achieved when x is 0. Another way of seeing these graphs is regarding D, E, F as the reflection of A, B, C in Fig. 1.

Problem 3. Show that this reflection is a reflection over the x-axis from the equations of graphs.

Problem 4. In Fig. 2, we see that F is sharper than E, and E is sharper than D. Which one is sharper? $y = -5x^2$ or $y = -x^2$?

Now, let's consider a general case. We consider $a > 0$ first. Completing the square, we get

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \tag{1}$$

Notice that the minimum value the term $a(x + \frac{b}{2a})^2$ can have is 0, which is achieved when $x + \frac{b}{2a} = 0$. Therefore, the minimum of (1) is $-\frac{b^2-4ac}{4a}$ and achieved when $x = -\frac{b}{2a}$. This point is called the "vertex." In other words,

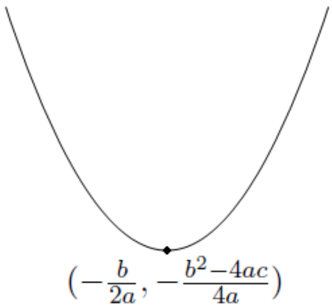

Figure 3: $y = ax^2 + bx + c, a > 0$

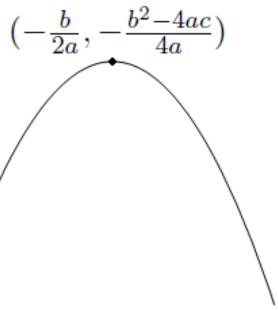

Figure 4: $y = ax^2 + bx + c, a < 0$

the vertex is at $(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a})$. See Fig. 3.

Problem 5. What is the minimum of $y = ax^2$, for $a > 0$? When is it achieved? What is the vertex of $y = ax^2$?

Problem 6. We can regard the graph $y = ax^2 + bx + c$ as the graph $y = ax^2$ shifted by a certain distance in positive or negative x-direction and another certain distance in positive or negative y-direction. What are these distances and directions? (Hint[1])

Now, let's consider the case $a < 0$. Completing the square, we get

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \tag{2}$$

just as (1). Notice that the maximum value the term $a(x + \frac{b}{2a})^2$ can have is 0, as a is negative, and this maximum is achieved when $x + \frac{b}{2a} = 0$. Therefore, the maximum of (1) is $-\dfrac{b^2 - 4ac}{4a}$, which is achieved when $x = -\dfrac{b}{2a}$. This point is called the "vertex." In other words, the vertex is at $(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a})$. See Fig. 4.

Summary

- For a graph $y = ax^2 + bx + c$ for $a > 0$, y increases as x goes to negative infinity or positive infinity. The y-minimum point is called the "vertex."
- For a graph $y = ax^2 + bx + c$ for $a < 0$, y decreases as x goes to negative infinity or positive infinity. The y-maximum point is called the "vertex."
- You can obtain the coordinate of vertex by completing the square.

[1]See (1), or consider the position of vertex.