Inscribed circle in a triangle

A triangle has three angles. See Fig. 1. We can bisect each angle, and the lines which bisect each angle meet at a point D. In other words, if we have $\angle ABD = \angle CBD$ (denoted with $\circ$) and $\angle ACD = \angle BCD$ (denoted with $\times$), we have $\angle BAD = \angle CAD$ (not denoted in the figure). This is so from the following reason. As there is only one line that bisects $\angle CAB$, if $\overline{DA}$ does so, it implies that D is indeed the point that the three bisected lines meet.

So, why do they meet at a single point? Why does $\overline{DA}$ bisect $\angle CAB$? To prove this, we will first draw three altitudes from D to each side. See Fig. 1 again. These altitudes are drawn in dotted lines. $\overline{DE}$ is perpendicular to $\overline{BC}$, $\overline{DG}$ is perpendicular to $\overline{AB}$, and $\overline{DF}$ is perpendicular to $\overline{AC}$.

Problem 1. Explain why $\triangle DBG$ is congruent to $\triangle DBE$. Is it SSS, SAS, or ASA?

Thus, we see that $\overline{DG} = \overline{DE}$. Similarly, $\triangle DCE$ is congruent to $\triangle DCF$. Therefore, $\overline{DE} = \overline{DF}$. Combining these two equations, we see that $\overline{DG} = \overline{DF}$. Now, notice that both $\triangle ADG$ and $\triangle ADF$ are right triangles. From the Pythagorean theorem, we have

$$\overline{AG} = \sqrt{\overline{AD}^2 - \overline{DG}^2}, \qquad \overline{AF} = \sqrt{\overline{AD}^2 - \overline{DF}^2} \tag{1}$$

Since $\overline{DF} = \overline{DG}$, we conclude $\overline{AG} = \overline{AF}$. Thus, $\triangle ADG$ and $\triangle ADF$ are SSS congruent. Therefore, in conclusion, we see that $\angle DAG = \angle DAF$, thus proving our assertion.

Final comment. As $\overline{DE} = \overline{DF} = \overline{DG}$, we can draw a circle of which center is D and passes the point E, F, G. In other words, its radius r is given by $r = \overline{DE} = \overline{DF} = \overline{DG}$. See Fig. 2. Notice also that $\overline{DE}$ is perpendicular to $\overline{BC}$. Thus, this circle meets $\overline{BC}$ at a single point, namely E. Similarly, it meets $\overline{CA}$ at a single point, namely, F. Similarly, it meets $\overline{AB}$ at a signle point, namely, G. We say that the circle is inscribed in the triangle.

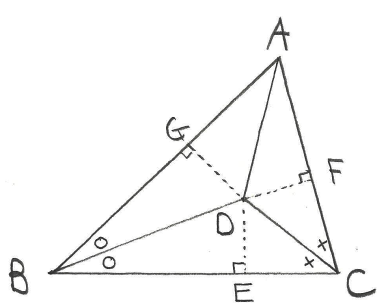

Figure 1: bisected angles meeting at a point

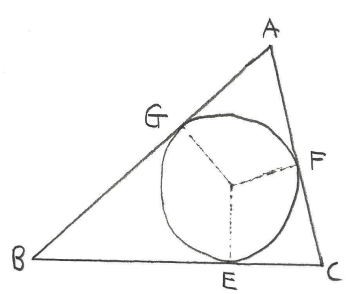

Figure 2: inscribed circle in a triangle

Summary

- Three lines that bisect each angle of triangle meet at a point.
- We can draw a circle, of which the center is such a point, and touches each side. We call such a circle "inscribed circle in triangle."